

A *DNR* Perspective on Mathematics Curriculum and Instruction Part II

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0 Introduction

This is the second in a series of two papers, the goal of which is to contribute to the debate on a pair of questions that are on the mind of many mathematics educators—teachers, teacher leaders, curriculum developers, and researchers who study the processes of learning and teaching—namely:

1. What is the mathematics that we should teach in school?
2. How should we teach it?

Clearly, a pair of papers is not sufficient to address these colossal questions, which are inextricably linked to other difficult questions—about student learning, teacher knowledge, school culture, societal need, and educational policy, to mention a few. My goal in these two publications is merely to articulate a pedagogical stance on these two questions. The stance is not limited to a particular mathematical area or grade level; rather, it encompasses the learning and teaching of mathematics in general.

This stance is oriented within a theoretical framework, called *DNR-based instruction in mathematics* (*DNR*, for short). The initials *D*, *N*, and *R* stand for three foundational instructional principles in the framework: *duality*, *necessity*, and *repeated reasoning*—to be discussed later in this paper. *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts*—constructs defined and oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning.

This paper begins, in Section 1, with *DNR* premises. Section 2 is a synopsis of Paper I. The reader is strongly encouraged to read Paper I before reading this paper. The two papers correspond roughly to the above two questions: Paper I focuses on the constituent elements of mathematics curricula, and this paper on constituent elements of mathematics teaching. However, since one's approach to teaching necessarily depends on one's views of learning, this paper devotes a portion of its space to the *DNR*'s definition of learning. Learning and teaching will be discussed in Sections 3 and 4, respectively. What does all this have to do with the topic of this Special Issue: *empirical research on mathematics teachers and their education*? The answer to this question will become clearer as the paper unfolds, especially in Section 5, when the concept of *teacher's knowledge base* is defined in terms of the constituent elements of learning and teaching laid out in the preceding sections (Sections 3 and 4). The paper concludes, in Section 6, with a comment on the relationships among *DNR*'s premises.

1 *DNR*'s Premises

A major effort was made to state the *DNR* underlying assumptions explicitly. These assumptions, called *DNR* premises, were not conceived a priori before *DNR* was formulated, but instead emerged in the process of reflection on and exploration of justifications for the *DNR* claims. Collectively, these premises underlie *DNR*'s philosophy of mathematics and the learning and teaching of mathematics.

DNR has eight premises. They are loosely organized in four categories:

1. Mathematics

- **Mathematics:** Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history.

2. Learning

- **Epistemophilia:** Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience.
- **Knowing:** Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.
- **Knowing-Knowledge Linkage:** Any piece of knowledge humans possess is an outcome of their resolution of a problematic situation.
- **Context-Content Dependency:** Learning is context and content dependent.

3. Teaching

- **Teaching:** Learning scientific knowledge (such as mathematics) is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance.

4. Ontology

- **Subjectivity:** Any observations humans claim to have made is due to what their mental structure attributes to their environment.
- **Interdependency:** Humans' actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions.

As the reader might have recognized, these premises—with the exception of the Mathematics Premise, which has been discussed broadly in Harel (2008) and in Paper I—are taken from or based on known theories. Briefly, the Epistemophilia Premise follows from Aristotle (Lawson-Tancred, 1998, Klein, 1992); the Adaptation Premise is the nucleus of Piaget's theory of equilibration (Piaget, 1985); the Learning-Knowledge Linkage Premise, too, is inferable from Piaget, and is consistent with Brousseau's claim that "for every piece of knowledge there exists a fundamental situation to give it an appropriate meaning" (Brousseau, 1997); the Context-Content Dependency Premise is consistent with modern cognitive theories of knowledge acquisition, according to which learning is contextualized; the Teaching Premise derives from Vygotsky's (1978) known *ZPD* (Zone of Proximal Development) idea; the Subjectivity premise and the Interdependency Premise follow from both Piaget's constructivism theory (see, for example, von Glasersfeld, 1983) as well as from information processing theories (see, for example, Chiesi, Spilich, & Voss, 1979; Davis, 1984).

Why are these premises needed? *DNR* is a conceptual framework for the **learning** and **teaching** of **mathematics**. As such, it needs lenses through which to see the realities of the different actors involved in these human activities—mathematicians, students, teachers, school administrators, etc., particularly the realities of the students as learners in different stages in their conceptual development. *DNR* also needs a stance on the nature of the targeted knowledge to be taught—mathematics—and of the learning and teaching of this knowledge.

Starting from the end of the premises list, the two Ontology Premises—Subjectivity and Interdependency—orient our interpretations of the actions and views of students and teachers. Implications of the ontological positions expressed by these premises to mathematics education are not new of course. Scholars such as von Glasersfeld, Leslie Steffe, and Patrick Thompson were notable pioneers in offering and implementing research and curricular programs rooted in these positions (see, for example, Steffe, Cobb, & Glasersfeld, 1988; Steffe & Thompson, 2000). These scholars articulated essential implications to mathematics curriculum and instruction: that students’ realities *are* their actual experiences, not what we speak of as observers; that when we describe our observations of students’ experiences we merely offer a model describing our conception of what we have observed; that for these models to be effective pedagogically, they should include students’ actions—what we see and hear—as well as their possible causes. All these elements are integral parts of *DNR*. As was discussed in Harel (2008), this subjective stance is already present in the definitions of the constructs “way of understanding” and “way of thinking,” and hence in the conceptualization of the Mathematics Premise.

The Mathematics Premise comprises its own category; it concerns the nature of the mathematics knowledge—the targeted domain of knowledge to be taught—by stipulating that ways of understanding and ways of thinking are the constituent elements of this discipline, and therefore instructional objectives must be formulated in terms of both these elements, not only in terms of the former, as currently is largely the case (see Paper I).

Each of the four Learning Premises—Epistemophilia, Knowing, Knowing-Knowledge Linkage, and Context-Content Dependency—attend to a different aspect of learning: The Epistemophilia Premise is about humans’ propensity to know, as is suggested by the term “Epistemophilia:” love of episteme. Not only do humans desire to solve puzzles in order to construct and impact their physical and intellectual environment, but also they seek to be puzzled. The term “puzzle” should be interpreted broadly; it refers to problems intrinsic to an individual or community, not only to recreational problems, as the term is commonly used. Such problems are not restricted to a particular category of knowledge, though here we are solely interested in the domain of mathematics. The Epistemophilia Premise attends to another significant issue. It claims that *all* humans are capable of learning if they are given the opportunity to be puzzled, create puzzles, and solve puzzles. While it assumes that the propensity to learn is innate, it rejects the view that individual differences reflect innate basic capacities that cannot be modified by adequate experience (social, emotional, psychological, and intellectual).

The Knowing Premise is about the mechanism of knowing—that the means—the only means—of knowing is a process of assimilation and accommodation. A failure to assimilate results in a disequilibrium, which, in turn, leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. In essence, this premise is the basis for the position, held by many scholars (e.g. Brownell, 1946; Davis, 1992; Hiebert, 1997; Thompson, 1985), that problem solving is the only means of learning.

The Context-Content Dependency Premise is about contextualization of learning. The premise does not claim that learning is entirely dependent on context—that knowledge acquired in one context is not transferrable to another context, as some

scholars (Lave, 1988) seem to suggest. This claim, which implies, for example, that abstraction is of little use, is obviously not true, as is convincingly argued by Anderson, Reder, & Simon (1996). Instead, the Context-Content Dependency Premise holds that ways of thinking belonging to a particular domain are best learned in the context and content of that domain. Consider, for example, reification—the way of thinking where one reconceptualizes processes as conceptual entities, objects the mental system can reason about in a direct way (Greeno, 1980). Beginning at infancy and throughout life, humans form conceptual entities through interaction with their physical and social environments. They effortlessly reason directly about perceptual and social concepts, such as “texture” and “color” and “friendship” and “justice,” directly without a need to unpack or replay the experiences that led to their construction. The ease in which reification is applied in one’s daily life does not, however, guarantee its successful application in other domains, particularly mathematics. It has been well documented that reifying processes into conceptual entities in mathematics is difficult. For example, reconceptualizing the concept of function from a mapping process into an object—as an element of a group or a vector space, for example—is difficult even for college students (Dubinsky, 1991; Sfard, 1991); reconceptualizing the concept of fraction from a multiplicative relation into a single number is difficult for middle-grade students; and reconceptualizing the process of counting into a number name is far from being trivial for early-grade students (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

There is a reason for both “context” and “content” in the formulation of the Context-Content Premise. We have already explained the role of “context” in learning. The inclusion of “content” is to emphasize that each mathematical content area is characterized by a unique set of ways of thinking (and ways of understanding obviously). For example, the set of ways of thinking that characterizes combinatorics is different from that that characterizes topology. Even within the same area, say Euclidian geometry, the ways of thinking that characterize plane geometry, for example, are not identical to those that characterize spatial geometry.

Finally, the Teaching Premise asserts that expert guidance is indispensable in facilitating learning of scientific knowledge. This premise is particularly needed in a framework oriented within a constructivist perspective, like *DNR*, because one might minimize the role of expert guidance in learning by (incorrectly) inferring from such a perspective that individuals are responsible for their own learning or that learning can proceed naturally and without much intervention (see, for example, Lerman, 2000). The Teaching Premise rejects this claim, and, after Vygotsky, insists that expert guidance in acquiring scientific knowledge—mathematics, in our case—is indispensable to facilitate learning.

The Teaching Premise leads naturally to the question concerning the necessary knowledge that an expert guide—a teacher—must possess and the nature of effective teaching practices that can bring about learning. The *DNR* constituent elements of teaching (Section 4) coupled with the definition of *teacher’s knowledge base* (Section 5) attend to this question.

2. Constituent Elements of Mathematics Curriculum: A Synopsis of Paper I

Paper I addresses the question, “What is the mathematics that we should teach in school?” *DNR*’s position on this question, based on its first premise, is that the constituent elements of mathematics, and therefore of desirable mathematics curricula,

are ways of understanding *and* ways of thinking. A way of understanding is a product of a mental act, whereas a way of thinking is a characteristic of ways of understanding associated with such an act. The triad “mental act,” “way of understanding,” and “way of thinking,” is central in *DNR*. It is a generalization of the triad “proving, proof, and proof scheme,” which emerged in investigations concerning the learning and teaching of mathematical proof (see, for example, Harel & Sowder, 1998, 2007). The generalization was necessitated by the realization that the processes of learning and teaching mathematical proof involve numerous mental acts, such as “interpreting,” “connecting,” “modeling,” “generalizing,” “searching,” and “symbolizing,” and so attention to proving alone is insufficient to identify and communicate classroom and clinical observations.

Examples of major categories of ways of thinking include problem-solving approaches, proof schemes, and beliefs about mathematics. Of these, the proof schemes category was taxonomized on the basis of students’ work and historical development (Harel and Sowder, 1998). The taxonomy consists of three classes—*external conviction*, *empirical*, and *deductive*—each with subclasses. An example of the empirical proof scheme, which will be attended to later in this paper, is the *result pattern generalization* (RPG) way of thinking, where one generalizes a finite pattern without attending to its underlying structure. The *process pattern generalization* (PPG) way of thinking, on the other hand, manifests one’s ability to reason in terms of underlying structures of numerical and geometric patterns. It is an instance of and precursor for the deductive proof scheme.

In a nutshell, Paper I argues that the answer to the first question in the opening of this paper should be driven by desirable ways of understanding *and* ways of thinking, not only by the former, as is currently the case. An important goal of research in mathematics education is, therefore, to identify these ways of understanding and ways of thinking; recognize, when possible, their development in the history of mathematics; and, accordingly, develop mathematics curricula and teacher education programs that aim at helping students construct them.¹ Section 4 deals with pedagogical tools needed for effectively teaching such curricula. These tools hinge, in part, upon *DNR*’s definition of *learning*, which is discussed in the next section.

3. Constituent Elements of Mathematics Learning

One’s view of learning might be informed and formed by a scholarly-based theory—such as behaviorism, information processing, or constructivism—or by an informal experience. In *DNR*, the definition of mathematics learning follows from the *DNR* premises. It follows from the Learning-Knowledge Linkage Premise that problem solving is the means—the only means—to learn. When one encounters a problematic situation, one necessarily experiences phases of disequilibrium, often intermediated by phases of equilibrium. Disequilibrium, or perturbation, is a state reached when one encounters an obstacle. Its cognitive effect is that it “forces the subject to go beyond his current state and strike out in new directions” (Piaget, 1985, p. 10). Equilibrium is reached when one perceives success in removing such an obstacle. In Piaget’s terms, it is a state when one modifies her or his viewpoint (accommodation) and is able, as a result,

¹ This should not be taken to imply that mathematics curricula should mirror psychological development of students. Rather, the assumption here is that historical developments can shed light on cognitive processes of learning and, in turn, help provide a perspective on teaching.

to integrate new ideas toward the solution of the problem (assimilation). But what constitutes perturbation? More relevant to this paper, what constitutes perturbation in mathematical practice? *DNR* defines perturbation in terms two types of human needs: *intellectual need* and *psychological need*—terms to be discussed in the next section. *DNR*'s definition of learning, thus, incorporates these two needs and, consistent with the Subjectivity Premise, it also incorporates the knowledge currently held and newly produced during the learning process. Further, since our interest is restricted to mathematics learning, this knowledge is defined in terms of ways of understanding and ways of thinking, by the Mathematics Premise. Thus, *DNR*'s definition of *learning* is:

Learning is a continuum of disequilibrium-equilibrium phases manifested by (a) intellectual *and* psychological needs that instigate *or* result from these phases and (a) ways of understanding *or* ways of thinking that are utilized *and* newly constructed during these phases.

It should be pointed out here, and elaborated on elsewhere (Harel & Koichu, in preparation), that this definition is operational in that it specifies the nature of change in knowledge and the stimuli that result from and instigate the change. In this respect, the italicized disjunctives (*or*) and conjunctives (*and*) in the definition are important.

3.1 Intellectual Need versus Psychological Need

As was explained earlier, the above definition incorporates two categories of interrelated human needs, *intellectual needs* and *psychological needs*, and two constructs of knowledge, *ways of understanding* and *ways of thinking*. The latter pair was discussed extensively in Harel (2008) and in Paper I. In this section we discuss the former pair.

Let K be a piece of knowledge possessed by an individual or community. By the Learning-Knowledge Linkage Premise, there exists a problematic situation S out of which K arose. S (as well as K) is subjective, by the Subjectivity Premise, in the sense that it is a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, her or his current knowledge. Such a problematic situation S , prior to the construction of K , is referred to as an individual's *intellectual need*: S is the need to reach equilibrium by learning a new piece of knowledge.² Perturbational states do not necessarily lead to knowledge construction—a person can remain in a state of disequilibrium either due to inability or lack of motivation. Here, however, we are talking about the case where a perturbational state S has led to the construction of K . In this case, if the individual also sees how K resolves S , then we say that the individual possesses an *epistemological justification for the creation of K*.

Epistemological justifications are not to be confused with *logical justifications*. A logical justification is a mathematical proof—a chain of arguments the mathematics community at large believes to be reducible to a *derivation*: a finite sequence of formulas in the language of a formalized theory T , each of which is either a logical axiom, an axiom of T , or is the result of applying one of the finitely many explicitly stated rules of inference to previous formulas in the sequence. Thus, while logical justifications concern

² There more to say about the “intellectual” part of “intellectual need.” Historical and epistemological analyses led to five categories of intellectual needs: *need for certainty*, *need for causality*, *need for computation*, *need for communication*, and *need for connection*. Unfortunately, due to space limitation, these categories of needs will not be discussed in this paper.

the “truth” of knowledge, epistemological justifications concern the genesis of knowledge, the perceived reasons for its birth in the eyes of the learner.

There is common confusion between *intellectual need* and *motivation*. The two are related but are fundamentally different. While intellectual need belongs to epistemology, motivation belongs to psychology. Intellectual need has to do with the birth of a disciplinary knowledge with people from their current knowledge through engagement in problematic situations conceived as such by them. Motivation, on the other hand, has to do with people’s desire, volition, interest, self determination, and the like. Indeed, before one immerses oneself in a problem, one must desire, or at least be willing, to engage in the problem, and once one has engaged in a problem, often persistence and perseverance are needed to continue the engagement. These characteristics are manifestations of *psychological needs*: motivational drives to initially engage in a problem and to pursue its solution. The existence of these needs is implied from the Epistemophilia Premise, which asserts that people desire to solve problems and to look for problems to solve—they do not passively wait for disequilibrium.

Modern theories of motivation argue, on the basis of a wide range of empirical studies, that such psychological needs exist and are innate. The Self Determination Theory (SDT, see, for example, Deci & Ryan, 1980, 1985, 1990) identifies three such needs: (a) *need for autonomy*—the need for freedom to follow one’s inner interest rather than being controlled by extrinsic rewards; (b) *need for competence*—the need for having an effect, for being effective in one’s interactions with the environment; and (c) *need for relatedness*—the need for a secure relational base with others. These needs, according to the SDT theory, are “*innate psychological nutriments that are essential for ongoing psychological growth, integrity, and well-being.*” (p. xx) The SDT acknowledges physiological drives (such as hunger, thirst, and sex) argued by Hull (1943) and Freud (1967) but gives primacy to psychological needs in its “exploration of issues such as human learning, interpersonal relations, and the general mastery and management of people’s physical and social environments.” Further, the SDT suggests that the drive-based behaviors that Hull and Freud described are regulated by psychological processes and therefore interface with the needs for autonomy, competence, and relatedness. These needs, according to SDT, are expressions of a fundamental human trajectory toward vitality, integration, and health, and this “organismic tendency will be actualized so long as the necessary and appropriate nutriments are attainable but will give way to the emergence of nonoptimal psychological outcomes under conditions of threat or deprivation.” Deci & Ryan (2000) cite studies indicating that evaluations and external rewards decrease creativity (Amabile, 1982), cognitive flexibility (McGraw & McCullers, 1979), and conceptual learning (Grolnick & Ryan, 1987). Deci, Koestner, & Ryan (2001) also show that tangible rewards (e.g., money, prizes, and trophies) have a substantial undermining effect on intrinsic motivation.

Despite the label “psychological needs,” these needs include what might be referred to as social-cultural needs. It goes beyond the scope of this paper to analyze this inclusion relation, but I will point to some of the social-cultural needs in the context of schooling: By and large, students accept the obligation to attend school to learn—an obligation that is rooted in the cultural values and social conventions of the society in which we live. This need manifests itself in different but interrelated ways. First, there is the need that originates from external expectation, explicit or implicit, by authoritative

figures—such as teachers and parents—and the society at large. This need is particularly dominant in current teaching practices and is utilized through a complex system of rewards and punishments (e.g., grades, contests, etc.). Second, there is the need driven by causes of self-advancement, such as advancing one's social stature and improving one's economic conditions. Third, there is the need that stems from a desire to advance societal causes, such technological, political, environmental, and social justice causes. Common to these types of need is a sense of a social obligation: to an authority, to oneself within a society, or to a society at large. Collectively, therefore, they might be referred to as *social-cultural needs*.

Psychological needs, thus, belong to the field of motivation, which addresses conditions that activate and boost—or, alternatively, halt and inhibit—learning in general. In contrast, intellectual needs refer to the epistemology of a particular discipline with an individual or community from the knowledge they currently hold. Of course, as human behaviors, the two categories of needs are related; in fact, they complement each other: On the one hand, knowledge of a discipline always stem from problematic situations unique to that discipline and understood as such by an individual or community studying the discipline. On the other hand, in suitable physical, emotional, and social environments, humans are ready to engage in these problematic situations and persevere in pursuing their solutions.

As was mentioned earlier, *DNR*'s definition of *mathematics learning* incorporates both students' intellectual needs and psychological needs, though *DNR* does not at this stage of its development attend to the latter. Consistent with the Subjectivity Premise, this definition also incorporates the knowledge currently held or newly produced during the learning process.

Research in mathematics education has offered useful models for learning trajectories of various mathematical concepts and ideas, but, as with *DNR*, psychological needs in the sense discussed here have not been a major focus. Examples of such models include: Fischebein's (1985) intuitive models for the concepts of multiplication and division; Tournaire's (1986) model for the concept of proportionality; Dubinsky's action-process-object-schema model for the concept of function (E. D. Dubinsky & McDonald, 2001); Schoenfeld's (1992) model for thinking mathematically. However, relative to the broad scope of our definition of learning, these models are largely partial. This is expected due to the enormous empirical and theoretical difficulties in building models that incorporate phases of disequilibrium-equilibrium, their utilized or resultant ways of understanding and ways of thinking, and the cognitive, social, and affective stimuli that result from and instigate the various phases. Such comprehensive models, however, are needed and the hope is that they will be constructible in the future.

4. Constituent Elements of Mathematics Teaching

This section discusses *DNR*'s three foundational instructional principles: *duality*, *necessity*, and *repeated reasoning*. The *duality principle* deals with the developmental interdependency between ways of understanding and ways of thinking; the *necessity principle* with students' *intellectual need*; and the *repeated reasoning principle* with internalization, organization, and retention of knowledge. As we will see, the three principles are inextricably linked. A single principle, taken individually and separately from the other two principles, is likely to be of a lesser pedagogical value than if is considered in the context of the other two.

4.1 Developmental Interdependency between Ways of Understanding and Ways of Thinking: The *Duality Principle*

Let's us begin by taking a closer look at the formation of the empirical proof scheme (see Paper I). Recall that with this scheme one proves—that is, removes doubts about the truth of an assertion—inductively—by relying on evidence from examples of direct measurements of quantities, substitutions of several numbers in algebraic expressions, etc—or perceptually—by relying on evidence from physiological senses such as visual or tactile perceptions). Research has shown that the empirical proof scheme is prevalent and persistent among students at all grade levels (Chazan, 1993; Goetting, 1995; Guershon Harel & Sowder, 2007). What might be the cause for the dominance of this scheme?

Students do not come to school as blank slates, ready to acquire knowledge independently of what they already know (Piaget, 1952, 1969, 1973, 1978). Rather, what students know now constitutes a basis for what they will know in the future. This is true for all ways of understanding and ways of thinking associated with any mental act; the mental act of proving is no exception. In everyday life and in science in general the means of justification available to humans are largely limited to empirical evidence. Since early childhood, when we seek to justify or account for a particular phenomenon, we are likely to base our judgment on similar or related phenomena in our past (Anderson, 1985). Given that the number of such phenomena in our past is finite, our judgments are typically empirical. Through such repeated experience, which begins in early childhood, our hypothesis evaluation becomes dominantly empirical; that is, the proofs that we produce—to ascertain for ourselves or to persuade others—become characteristically inductive or perceptual. If during early grades our judgment of truth in mathematics continues to rely on empirical considerations, the empirical proof scheme will likely dominate our reasoning in later grades and more advanced classes, as research findings clearly show. While unavoidable, the extent of the dominance of the empirical proof scheme on humans is not uniform. Children who are raised in an environment where sense making is encouraged and debate and argumentation is an integral part of their social interaction with adults are likely to have a smoother transition to deductive reasoning than those who are not raised in such an environment.

A simple, yet key, observation here is this: the proofs children produce to prove assertions and account for phenomena in everyday life impact the kind and robustness of the proof schemes they form. Proofs, as was explained earlier, are ways of understanding associated the mental act of proving, and proof schemes are ways of thinking associated with the same mental act. Hence, a generalization of this observation is: for any mental act, the ways of understanding one produces impact the quality of the ways of thinking one forms.

Of equal importance is the converse of this statement; namely: For any mental act, the ways of thinking one has formed impact the quality of the ways of understanding one produces. The latter statement is supported by observations of students' mathematical behaviors; for example, when proving. As was indicated earlier, this scheme does not disappear upon entering school, nor does it fade away effortlessly when students take mathematics classes. Rather, it continues to impact the proofs students produce. It takes enormous instructional effort for students to recognize the limits and role of empirical evidence in mathematics and begin to construct alternative, deductively-based proof

schemes. Even mathematically able students are not immune from the impact of the empirical proof scheme (see Fischbein & Kedem, 1982). Students' past mathematical experience, however, plays a critical role in the extent to which their empirical proof schemes impact the proofs they produce.

This analysis points to a reciprocal developmental relationship between ways of understanding and ways of thinking, which is expressed in the following principle:

The *Duality Principle*:

1. T_U : **Students develop ways of thinking through the production of ways of understanding, and, conversely:**
2. U_T : **The ways of understanding they produce are impacted by the ways of thinking they possess.**

For easier reference, the first statement of the Duality Principle is denoted by T_U (indicating that ways of understanding serve as a conceptual basis for the development of ways of thinking), and its converse by U_T (indicating that ways of thinking serve a conceptual basis for the production of ways of understanding).

The analysis preceding the Duality Principle, and hence the principle itself, is implied from the Interdependency Premise. To see this, one only need to recognize that a person's ways of thinking is part of her or his view of the world, and that a person's ways of understanding are manifestations of her or his actions. Specifically, the U_T statement is an instantiation of the premise's assertion that humans' actions are induced and governed by their views of the world, whereas the T_U statement is an instantiation of the premise's assertion that humans' views of the world are formed by their actions.

4.2 Intellectual Need: The Necessity Principle

When we talk about problems in the context of mathematics curricula, we refer to learning-teaching events that involve two sets of interpretations (by the Subjectivity Premise): the set that belongs to the poser of the problem and the set that belongs to the one to whom the problem is posed. An important consequence of this simple observation is that the two sets are not always identical and, in many cases, do not even intersect. Teachers might present a problem to their class and incorrectly assume that their students share their interpretation(s) of the problem. Conversely, students might pose a question to their teacher, who either encounters difficulty understanding what they are asking or interprets their question differently from how it is intended. Of particular interest are the scenarios where what is conceived as a problem by the teacher is unproblematic to the students, and vice versa. In general, a situation that is problematic to one person but is unproblematic to another is referred to as *intrinsic* to the first and *alien* to the second. Accordingly, mathematics problems in a learning-teaching setting are of four categories: Either alien or intrinsic to both the student and the teacher (two categories) or alien to one or intrinsic to the other (two categories). The situation where a problem statement is intrinsic to both—is the only desired category among the four. As long as the problem is intrinsic to both, significant learning is likely to occur, even when—or perhaps especially when—the teacher's interpretation is different from those of the students.

Unfortunately, none of the other three cases is uncommon. When teachers lack understanding of the mathematics they teach—a not uncommon phenomenon (see Ball, 1991; Cohen, 1990; G. Harel, Behr, Post, & Lesh, 1992; Ma, 1999; Simon, 1993)—what

they teach is usually *alien* to them and, consequently, alien to their students. Consider the following episode: A ninth-grade teacher, teaching algebra and geometry, requires his students to accompany each assertion written on the left-hand side of two-column proofs by a reason on the right-hand column. For example, students in his geometry class justified the assertions, “ $AB \cong AB$ ” by the phrase “reflexive property;” the assertion “If $\angle ABC = 30^\circ$ and $\angle CBD = 45^\circ$, then $\angle ABD = 75^\circ$,” by “additive property.” Similarly, students in his algebra class justified the assertion “ $a + b = b + a$ ” by the phrase “commutative property,” “ $(a + b) + c = a + (b + c)$ ” by “associative property,” and “ $(-1)b = -b$ ” by “multiplying by -1 property.” It turned out that the teacher and his students viewed these assertions as obvious—ones that require no justification—but all felt compelled to follow rules: the students had to follow those imposed by their teacher, and the teacher those imposed by the textbook. Thus, the task to justify was alien to the teacher and to his students.

Astonishingly, the occupation with such properties is not uncommon even in elementary mathematics. The requirement to justify operations on real numbers in terms of basic properties such as “commutativity,” “associativity,” and “identity” is not exclusive to the secondary school mathematics; it is also common in elementary mathematics. Here, too, the task to justify is commonly *alien* to both the teachers and students. For example, a fifth-grade teacher assigned the problem:

Use properties to find n in the following equations: (1) $55 + 8 = n + 55$, (2)
 $8 + (2 + 3) = (n + 2) + 3$, “ $17 + 0 = n$.

The properties referred to in this assignment are the commutative, associative, and identity properties. Students were expected to solve the three problems by resorting to these three properties, respectively. The attention of many of these fifth graders was focused solely on teacher’s demand to use these properties rather than on the quantitative meaning of the equations. There were students who solved each of these problems directly (e.g., in Problem 1, some students first added 55 and 8 to get 63, and then looked for and found a number whose sum with 55 is 63), and then accompanied their answer by the property they somehow knew to be the one expected by the teacher (“commutative property,” in Problem 1). From the students’ point of view, thus, the task to use the properties to find the unknown n is likely to have been alien—merely to satisfy the teacher’s will—rather than intellectually intrinsic—to solve a problem they find intellectually puzzling. The teacher’s justification for the task she assigned, too, was alien: “So that students will do well be tested on these properties in future standardized tests.”

The lack of attention to students’ intellectual need is not restricted to proving; rather, it typifies the mathematics curricula in all subject and all grade levels. Consider the following two examples, one from secondary school algebra and the second from undergraduate mathematics (linear algebra):

After learning how to multiply polynomials, high-school students typically learn techniques for factoring (certain) polynomials. Following this, they learn how to apply these techniques to simplify rational expressions. Judging from the students’ perspective, the tasks of multiplying and factoring polynomials and simplifying rational expressions are alien, intellectually purposeless. They learn to transform one form of expression into another form of expression without a clear understanding of the mathematical purpose such transformations serve and the circumstances under which one form of expression is

more advantageous than another. A case in point is the way the quadratic formula is taught. Most algebra textbooks present the quadratic formula before the method of completing the square. Seldom do students see an intellectual purpose for the latter method—to solve quadratic equations and to derive a formula for their solutions—rendering completing the square problems *alien* to most students.

Typically, linear algebra textbooks introduce the pivotal concepts of “eigenvalue,” “eigenvector,” and “matrix diagonalization” with statements such as the following:

The concepts of “eigenvalue” and “eigenvector” are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form DX^{-1} , where D is diagonal. The latter factorization would provide important information about A , such as its rank and determinant.

The concepts of “eigenvalue” and “eigenvector” are needed to deal with the problem of computing a higher power of a given matrix, to study the long-term behavior of linear systems.

The concepts of “eigenvalue” and “eigenvector” are needed to deal with a problem that arises frequently in application of linear algebra—that of finding values of a scalar parameter λ for which there exists $x \neq 0$ satisfying $Ax = \lambda x$, where A is a square matrix.

In this section we consider the problem of factoring an $n \times n$ matrix A into a product of the form DX^{-1} , where D is diagonal. We will give necessary and sufficient condition for the existence of such a factorization and look at a number of examples. We begin by showing that eigenvectors belonging to distinct eigenvalues are linear independent.

Each of these introductory statements aims at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university mathematician), it is most likely to be alien to the students. It is so because a regular undergraduate student in an elementary linear algebra course is unlikely to realize from such statements the nature of the problem indicated, its mathematical importance, and the role the concepts to be taught (“eigenvalue,” “eigenvector,” and “diagonalization”) play in determining its solution.

What all these examples demonstrate is that the *intellectual need* component in (the *DNR* definition of) learning is largely ignored in teaching. Recall according to this definition, learning necessarily involves a continuum of disequilibrium-equilibrium phases instigated by intellectual need. The Necessity Principle attends to the indispensability of intellectual need in learning

The ***Necessity Principle***:

For students to learn the mathematics we intend to teach them, they must have a need for it, where ‘need’ here refers to *intellectual need*.

4.3 Internalization of Knowledge: The Repeated Reasoning Principle

Even if ways of understanding and ways of thinking are intellectually necessitated for students, teachers must still ensure that they internalize, retain, and organize this

knowledge. Repeated experience, or practice, is a critical factor in achieving this goal. Cooper (1991) demonstrated the role of practice in organizing knowledge. DeGroot (1965) concluded that increasing experience has the effect that knowledge becomes more readily accessible, with reduced effort to infer it: “[knowledge] which, at earlier stages, had to be abstracted, or even inferred, [is] apt to be immediately perceived at later stages.” (p. 33-34). Repeated experience results in fluency, or effortless processing, which places fewer demands on conscious attention. “Since the amount of information a person can attend to at any one time is limited (Miller, 1956), ease of processing some aspects of a task gives a person more capacity to attend to other aspects of the task (LaBerge and Samuels, 1974; Schneider and Shiffrin, 1985; Anderson, 1981, 1982; Lesgold et al., 1988)” (quote from NRC, 1999, p. 32)

The emphasis of *DNR-based instruction*, however, is on repeated reasoning that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization—a mental state that manifests itself in the ability to apply knowledge autonomously and spontaneously. The sequence of problems given to students must continually call for thinking through the situations and solutions, and problems must respond to the students’ changing *intellectual needs*. This is the basis for the *repeated reasoning principle*.

The *Repeated Reasoning Principle*: Students must practice reasoning in order to internalize ways of understanding and ways of thinking.

The repeated reasoning principle is complementary to the other two principles in that its aim is for students to internalize the ways of understanding and ways of thinking acquired through the application of the other two principles. Through repeated reasoning in solving intrinsic problems, acquired knowledge becomes autonomous and spontaneous. The principle emphasizes *repeated reasoning* that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is important for internalizing and consolidating concepts. The sequence of problems must continually call for thinking through the situations and solutions; thus, they must respond to the students’ changing intellectual needs.

5. Teacher’s Knowledge Base

An educational system can be thought of as a triad of agents together with an *action theory*. The agents are students, teachers, and institutions, such as school, school district, home, etc. The action theory consists of these agents’ shared meanings for “knowledge,” “learning,” and “teaching” and shared perspectives on the social, cultural, behavioral, and emotional factors involved in the learning and teaching of particular knowledge. Action theories are not usually explicit to their associated agents, though they determine the agents’ conceptions and shape and govern their actions. For example, an action theory dictates the responsibility and school-related behaviors of students, teachers, school administrators, and parents. It includes conventions for how learning occurs and what facilitates or impedes learning. It also offers tools for measuring what students know and sets expectations for what they ought to know. Thus, action theories orient educational systems as to what, why, and how to carry out school-related actions. A crucial aspect of any action theory is its assumptions. In most cases, these assumptions are not explicit, not even to their agents. It is necessary, however, to understand action

theories, for they can help us explain—perhaps even predict—educational systems’ behaviors, which, in turn, can help us improve their efficacy.

In this section we are particularly concerned with teachers’ action theories, not those belonging to educational systems in general, though the two are strongly related. Some crucial components of a teacher’s action theory comprise her or his *knowledge base*. Building on Shulman’s (1986; 1987) work and consistent with the views of other scholars (e.g., Brousseau, 1997; Cohen & Ball, 1999, 2000), *teacher’s knowledge base* was defined in Harel (1993) in terms of three components: *knowledge of mathematics*, *knowledge of student learning*, and *knowledge of pedagogy*. In *DNR* these terms are defined as follows:

- *Knowledge of mathematics* refers to a teacher’s *ways of understanding and ways of thinking*.
- *Knowledge of student learning* refers to a teacher’s conception of how students learn mathematics, in general, and how students learn a specific mathematical content, in particular.
- *Knowledge of pedagogy* refers to a teacher’s conceptions of how instruction facilitates or impedes learning. These conceptions include the teacher’s repertoire of instruction principles.

What is crucial here is the content of the three components comprising a teacher’s knowledge base, not their label, which may resemble those offered by others (e.g. Ball & Bass, 2000). In Harel (1993), the content of these three components was described in general, less precise terms, though undoubtedly was influenced by the *DNR* ideas, which at the time were neither explicit nor well-formed. Here, in contrast, the content of a teacher’s knowledge base is explicitly implied from *DNR*’s premises, concepts, and instructional principles. Of paramount importance is the question of how to develop an effective knowledge base for future and inservice teachers. Before we demonstrate *DNR*’s approach to teacher professional development, let’s discuss first relationships among the three components of the teacher’s knowledge base.

5.1 Teacher’s Knowledge Base: Relationship among Its Components

In *DNR*, a teacher’s knowledge of pedagogy and student learning rests on the teacher’s knowledge of mathematics. That is to say, although each of the three components of knowledge is indispensable for quality teaching, they are not symmetric: the development of a teachers’ knowledge of student learning and of pedagogy depends on and is conditioned by their knowledge of mathematics. An implication of this view to teacher education is that teachers’ knowledge of how student learn mathematics and how, accordingly, teach mathematics evolves, in part, in the development of and reflection on their own mathematical knowledge. To illustrate, consider the following episode from a two-year, on-site professional development study aimed at investigating the evolution of teachers’ knowledge base. One of the findings of this study is that teachers’ appreciation for students’ active participation in the development of a solution to a mathematical problem is a function of the quality of the teachers’ mathematical knowledge. For example, Lisa, one of the teacher participants in this study, developed and enthusiastically implemented an instructional activity where her tenth-grade class gradually discerned the formula for the sum of the interior angles in a convex polygon along with a mathematically acceptable justification for it. In telling the researchers about her experience, she pointed out, with great satisfaction and a sense of accomplishment,

that the students developed the formula on their own. On the other hand, Lisa, who had insufficient understanding of graphical representation of solutions to systems of linear inequalities, struggled to see the benefit of a multi-stage instructional activity that was designed to involve students in developing an understanding of how to solve and graph the solution of such systems. She decided, instead, to provide the students with a prescribed procedure of how solve such systems. Overall, Lisa's knowledge of pedagogy and of student learning seems to develop and grow out of self reflection on her knowledge of mathematics, not out of the institutional demand to improve her students' mathematical performance.

This position is implied from the Context Dependency Premise, according to which ways of thinking belonging to a particular discipline are best learn in contexts of that discipline. The discipline under consideration here is *mathematics learning and teaching*, what in the US is commonly called *mathematics education*, or *didactique*, in France.

The interrelationship among the three components of the teacher's knowledge base was explored by many studies. These studies point to the significance of the three components of a teacher's knowledge base: knowledge of mathematics, student learning, and pedagogy. Although these did not use the *DNR* definitions—nor is it claimed that their conceptual frameworks are consistent with *DNR*—it is possible to make a few conjectures as to the nature of the interrelationships among these three components of knowledge.

Some of these studies have shown that teachers with weak knowledge of mathematics teach primarily how to follow a procedure to solve a category of problems, give little or no assistance to their students in developing an understanding of why the procedure works (Ball, 1991), and fail in their attempts to provide clear explanation and justification for the procedure they present (Borko et al., 1992). Druva and Anderson (1983) showed that teachers with greater content knowledge and with more teaching experience were more likely to ask higher level, cognitively based questions. Teachers with more content knowledge sought to learn more about students' thinking through questioning and visceral discussion in their teaching compared to teachers with less content knowledge. McDiarmid et al. (1989) concluded that teachers' content and pedagogical knowledge critically influence the quality of their teaching; it determines what questions the teachers pose, what tasks they select, how they evaluate their students' understanding, and what curricular decisions they make. Mullens, Murnane, Willett (1996) showed that there is a positive relationship between teachers' mathematical knowledge and their students' learning of advanced mathematical concepts. Brown and Borko (Borko et al., 1992) concluded that teaching mathematics from a conceptual perspective is highly improbable if teachers lack deep conceptual understanding of the subject matter they are teaching. Recent work by Ma (1999) supports these findings. Ma has shown that compared to their counterparts in the United States, Chinese teachers have a more profound understanding of mathematics, which allows them to promote mathematical learning effectively. The superior performance of the Chinese teachers is especially marked in situations where students raise ideas that are outside the scope of the content being taught in class. Wiley and Yoon's (1995) study demonstrated that when teachers are involved in professional developments that emphasized the study of mathematics curriculum and instruction, their students' achievement increases. Similarly,

Brown, et al. (1995) reported that the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) study found higher achievement among students whose teachers were involved in a program that aimed at enhancing teachers' understanding of teaching strategies, having teachers implement new teaching strategies, and encouraging teachers to reflect on instructional outcomes. Similarly, Grouws and Schultz (1996) reported that studies conducted to evaluate the impact of the CGI (Cognitively Guided Instruction) research found that providing teachers with knowledge of how students think and opportunities to develop strategies in specific content domains changed teaching behaviors and improved student learning.

5.2 *DNR*-Based Activities with Teachers

DNR-based activities with teachers aim at providing the teachers with an intense exposure to *DNR*-based instruction, in both their roles as learners of mathematics and as teachers reflecting upon the mathematics they are learning. Typically, a *DNR* activity begins with a problem, in some cases a problem that the teachers had worked on the previous day or that had been assigned to them as homework. The teachers are free to work on each problem individually or in small groups. In most cases, they spend some time individually establishing an initial solution approach before they get into their small working groups. Each classroom activity consists of a subset of the following teaching segments: Teachers work on the problem individually and in small groups; representatives of the small groups present to the class the groups' solutions or attempts; reflection on and discussion of the thought processes of solution approaches, particularly difficulties the teachers encountered as they attempted to solve the problem; a whole-class discussion to discern differences and similarities of the different solutions; articulation of ways of understanding and ways of thinking necessitated by the problem; whole-class discussion of possible difficulties students might have with different elements in the problem or in its solution, including the sources of such difficulties; and discussion of actual work done by students in their class and possible instructional treatments to deal with students' difficulties.

The goal of rest of this section is threefold: (a) to illustrate a few of these teaching segments, (b) to specify the instructional objectives they intend to achieve, and (c) to point to their rationale in terms of *DNR*. To this end, I will discuss four actual classroom activities:

5.2.1 Advancing Ways of Understanding and Ways of Thinking

Problem: Two pipes are connected to a pool. One pipe can fill the pool in 20 hours, and the other in 30 hours. How long will it take the two pipes together to fill the pool?

After solving this problem and discussing their solution approaches, the teachers were presented with the following actual four categories of solutions provided by a ninth-grade class, and were asked to analyze the possible conceptual basis for these solutions.

Solution 1: Divide the pool into five equal parts. The first pipe would fill one part in four hours, and the second pipe in six hours. Hence, in 12 hours the first pipe would fill $\frac{3}{5}$ of the pool and the second pipe the remaining $\frac{2}{5}$.

Solution 2: It will take the two pipes 50 hours to fill the pool.

Solution 3: It will take the two pipes 10 hours to fill the pool.

Solution 4: It would take x hours. In one hour, the first pipe will fill $1/20$ of the pool whereas the second will fill $1/30$. In x hours the first pipe would fill $x/20$, and the second $x/30$. Thus, $x/20 + x/30 = 1$ (which they then solved to obtain $x = 12$.)

The teachers discussed these solutions and concluded that they represent different ways of understanding the given problem and that the way of thinking “Look for a key word” might have governed the way of understanding expressed in Solution 2 due the word “together” in the problem text. They also hypothesized that although Solution 3 is incorrect, it might indicate a qualitative realization by the students who offered it that the time needed to fill the pool by the two pipes must be less than that the time needed to fill the pool by one pipe. Of particular interest to the teachers was Solution 1, which was given by only one student, G. The discussion among the teachers centered on a possible juxtaposition of ways of thinking that seemed to have driven G’s solution, which may include “Draw a diagram,” “Guess and check,” and “Look for relevant relationships among the given quantities.”

This episode illustrates our goal to help the teachers realize that the approach a student chooses to solve a problem depends on how he or she represents, or interprets, the problem statement. The ways of thinking, “A problem can have multiple solutions” and “It can be advantageous to solve a problem in different ways,” are extended in our instructional interventions to concepts; namely, “A concept can have multiple interpretations” and “It can be advantageous to possess multiple interpretations of a concept.” These ways of thinking, although essential in the learning and creation of mathematics, are often absent from teachers and students’ repertoires of reasoning. To construct these ways of thinking, the teachers engaged in activities aimed at promoting multiple ways of understanding concepts. For example, through their solutions to different problems the teachers learned that the concept of fraction, say $3/4$, can be understood in different ways and it is advantageous to understand it in different ways. Such ways of understanding include: *unit fraction* ($3/4$ is the sum, $1/4 + 1/4 + 1/4$); *partition* ($3/4$ is the quantity that results from dividing 3 units into 4 equal parts); *measurement* ($3/4$ is the measure of a 3 cm long segment with a 4-cm unit ruler); *solution to an equation* ($3/4$ is the solution to $4x = 3$); *part-whole* ($3/4$ is 3 out of 4 units). Similarly, the teachers solved problems through which they learned multiple ways of understanding the symbol $y = f(x)$ and the significance of each interpretation. For example, they saw how one can understand $y = 6x^2 - 5$ in terms of a condition on the variables x and y —the set of all ordered pairs (x, y) for which y is equal to the quantity $6x^2 - 5$. While here $y = 6x^2 - 5$ is viewed as an equation, one can understand it as a function: for each input of x there corresponds the output $6x^2 - 5$. These newly acquired ways of understanding by the teachers were in turn utilized to enhance their knowledge of pedagogy and student epistemology. This was done by contrasting these mature interpretations with the interpretation commonly possessed by students. For example, through actual work of students, the teachers saw that for many students symbols such as $y = 6x^2 - 5$ represent no quantitative reality, except possibly that the “equal” sign is understood as a “do something signal,” where one side of the equation is reserved for the operation to be carried out and the other side for its outcome, as was documented by Behr, Erlwanger, & Nichols (1976).

5.2.2 Necessitating Ways of Understanding and Ways of Thinking

Problem: Take a fraction whose numerator and denominator are integers. Add a number different from zero to the numerator and divide the denominator by that number. When would the new fraction be greater than the original fraction?

In accordance with the necessity principle, the different interpretations of concepts were necessitated for the teachers by problems they solved, as the following discussion illustrates: The teachers discussed in small groups the approaches they took, their partial solutions, and their solutions. After 35 minutes, one of the teachers presented her group's solution. Her solution process involved many quadratic inequalities. During her presentation there were numerous questions and suggestions from the class regarding the solution process.

Following this presentation, one of the teachers, K, suggested checking a few cases to see if they agree with the final answer. ("Evaluate your solution" is a thinking strongly stressed throughout the interventions.) After about five minutes of group work, K indicated that he noticed a strange phenomenon: when different forms of the same fraction are used different results are obtained. He demonstrated his observation with the equivalent fractions $-2/5$ and $-6/15$. Indeed, for $-2/5$ any value of x works but for $-6/15$ only $x < 3 - \sqrt{3}$ or $x > 3 + \sqrt{3}$ work. This provided a great opportunity—intended and anticipated—for the instructor to point to the necessity to differentiate between "fraction" and "rational number". Although the ratio is the same, the fraction is not, and the solution with a different fraction, even if the ratio is the same, is also different. The solution for the fraction $-2/5$ is different than the solution for the fraction $-6/15$. This result was particularly amazing for the teachers.

5.2.3 Internalizing Ways of Understanding and Ways of Thinking

Problem: Prove the quadratic formula.

Prior to this problem, the teachers had repeatedly worked with many quadratic functions, finding their roots by essentially completing the square. They abstracted this process to develop the quadratic formula. In doing so they repeatedly transformed a given equation $ax^2 + bx + c = 0$ into an equivalent equation of the form $(x+T)^2 = L$ for some terms T and L , in order to solve for x (as $-T + \sqrt{L}$ and $-T - \sqrt{L}$). To get to the desired equivalent form, they understood the reason and need for dividing through by a , bringing c/a to the other side of the equation, and completing the square. For these teachers, the symbolic manipulation process was goal oriented and conditioned by quantitative considerations; namely, transformations are applied with the intention to achieve a predetermined intrinsic goal. In this case, the teachers practiced the way of thinking of transforming an algebraic expression into a desired form without altering its original quantitative value. This way of thinking—which is one characteristic of algebraic reasoning—was known among the teachers as the changing-the-form-without-changing-the-value habit of mind. We see here the simultaneous implementation of the duality principle, the necessity principle, and the repeated reasoning principle. In particular, the repeated application of this habit of mind helped the participant teachers internalize it, whereby they become autonomous and spontaneous in applying it.

5.2.4 Institutionalizing Ways of Thinking

Problem: Is it true that if a positive integer is divisible by 9 then the sum of its digits is divisible by 9?

Two of the solutions offered by the teachers were:

Answer 1: Yes, it is true, because I took many cases, and in each case when the number is divisible by 9, the sum of its digits is divisible by 9.

Answer 2: For 867, $867 = 8 \times 100 + 6 \times 10 + 7 \times 1$, which is $(8 \times 99) + (6 \times 9) + (8 + 6 + 7)$. Each of the first two addends, 8×99 and 6×9 , is divisible by 9, so the third addend, $8 + 6 + 7$, which is the sum of number's digits, must be divisible by 9. The same can be done for any number, so it is true that if the number is divisible by 9, the sum of its digits is divisible by 9.

In our earlier research (Harel & Sowder, 1998; Martin & Harel, 1989), Justification 1 was found to be the most common among college students. A probe into the reasoning in Justification 1 has revealed that those subjects' conviction stems from the fact that the proposition is shown to be true in a few instances, each with numbers that are supposedly randomly chosen—a manifestation of the *empirical proof scheme*, which is one of the most ubiquitous proof schemes held by students. With this proof scheme, students ascertain for themselves and persuade others about the truth of a conjecture by direct measurements of quantities, numerical computations, substitutions of specific numbers in algebraic expressions, etc.

In discussing these answers with the teachers, the goal was to help them internalize a critical distinction between the two: In Justification 2, learners generalize from a *process pattern*, in the other from a *result pattern*. In *process pattern generalization*, learners focus on regularity in the process; whereas in *result pattern generalization*, they focus on regularity in the result. Process pattern generalization is a way of thinking in which one's conviction is based on regularity in the process, though noticing regularity in the result might stimulate it. This behavior is in contrast to *result pattern generalization*, where proving is based solely on regularity in the result—obtained by substitution of numbers, for instance. Through repeated discussion of this fundamental difference in the context of mathematical problems, we aimed at helping the teachers refine their ways of thinking about what constitutes justification in mathematics—their proof schemes: from proof schemes largely dominated by surface perceptions, non-referential symbol manipulation, and proof rituals, to a proof scheme that is based on intuition, internal conviction, and ultimately logical necessity.

6 Final Comment

In his book, *Theory of Didactical Situation in Mathematics*, Brousseau raises the question whether theories and methods specific the learning and teaching of mathematics are needed since general theories and methods about learning and teaching do exist:

Does knowledge imported from fundamental disciplines itself allow, independently and without modification, the explanation of teaching phenomena and the production in a controlled manner of the desired modifications? Must we, on the contrary, create new concepts, a file of knowledge and related methods in order to study didactical situations? (Brousseau, 1997, p. 23)

Brousseau's answer to this question is unequivocally affirmative:

... only the global study of situations presiding over the manifestation of knowledge allows us to choose and connect knowledge from different origins; knowledge necessary to understand the subject's cognitive activities, as well as the item of knowledge she uses and the way in which she modifies it.

[Furthermore, this] study of (didactical) situations must in the end allow the derivation or modification of the necessary concepts currently imported from other scientific fields.

DNR is entirely consistent with this view, in that although it rests on concepts, themes, and findings from different theories, it is a conceptual framework that emerged from and addresses problems in one domain, that concerning the learning and teaching of mathematics. As the reader might have observed, seven out of the eight *DNR* premises—Premises 2-8—are not unique to mathematics. Only Premise 1 is about the content of mathematics. This is critical because *DNR*'s concepts and claims emerged from, and therefore must be restricted to, considerations concerning the application of these seven premises to the content defined by the first premise: the mathematical ways of understanding and ways of thinking that have evolved throughout history.

Of course, there remains the question: Why were these particular seven premises chosen over others? For example, why is the Piagetian Theory of Equilibration and not Behaviorism or Socio-Culturalism used as the orientation for a conceptual framework for the learning and teaching of mathematics? Any attempt to address this question in depth would take us far afield, away from the main object of this paper. A simple answer to this complex question is that the choice of these premises was the result of various experiences, formal and informal, coupled with analytic judgments as to the merits of the different theories about learning and teaching, in general, and the learning and teaching of mathematics, in particular.

Another important question concerns connections among the different premises. To begin with, the eight *DNR* premises are not logically independent. For example, the Subjectivity and Interdependency Premises are clearly related; one may consider the former as a corollary of the latter, and vice versa. As manifestations of humans' activities, the premises are strongly related. The relations among some are more conspicuous than among others, as in the following examples:

The Epistemophilia Premise is strongly connected to the Learning-Knowledge Linkage Premise, in that the claim that the only means of knowing by humans is problem solving implies that solving puzzles and seeing puzzles to solve are means of survival and hence natural humans' propensities.

The Mathematics Premise and the Content Dependency Premise, too, are related, although their relation is less explicit. Basic to the definitions of "way of understanding" and "way of thinking," and therefore to the Mathematics Premise, is the primary notion of "mental act" (see Harel, 2008). As acts of reasoning, mental acts are universal; that is, they are carried out in varying degrees by people in everyday and professional life. The acts of interpreting, conjecturing, justifying, abstracting, and generalizing, for example, are not unique to mathematics or science; rather, people interpret, conjecture, justify, abstract, and generalize, in every area of their everyday and professional life. However, one is unlikely—by the Content Dependency Premise—to develop mathematical ways of thinking in a context other than mathematics, and, conversely, the mathematical ways of thinking one develops may not be applicable in contexts other than mathematics. Thus,

the Content Dependency Premise expresses a particular position on transfer of knowledge: that knowledge does not transfer effectively between domains. The premise does not imply, however, that people do not attempt to apply ways of thinking they acquire in one domain when dealing with issues they encounter in another domain.

In sum, in formulating the *DNR* premises, the concern was not logical independency but usability in justifying *DNR*'s concepts and claims about the learning and teaching of mathematics.

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