

***DNR Perspective on Mathematics Curriculum and Instruction Part I:
Focus on Proving***¹

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A *DNR* Perspective on Mathematics Curriculum and Instruction Part 1: Focus on Proving

0 Introduction

This is the first in a series of two papers whose goal is to contribute to the debate on a pair of questions that are on the mind of many mathematics educators—teachers, teacher leaders, curriculum developers, and researchers who study the processes of learning and teaching—namely:

1. What is the mathematics that we should teach in school?
2. How should we teach it?

This paper addresses the first question, and the second paper, to appear in the next issue of *ZDM*, addresses the second question.

Clearly, a pair of papers is not sufficient to address these colossal questions, which are inextricably linked to other difficult questions—about student learning, teacher knowledge, school culture, societal need, and educational policy, to mention a few. My goal in these two publications is merely to articulate a pedagogical stance on these two questions. The stance is not limited to a particular mathematical area or grade level; rather, it encompasses the learning and teaching of mathematics in general. However, in this paper, the discussion of this stance is restricted to the first question and is instantiated mainly in proof-related contexts.

This pedagogical stance is oriented within a broader theoretical framework, called *DNR-based instruction* in mathematics (*DNR*, for short). The initials *D*, *N*, and *R* stand for three foundational instructional principles in the framework: *duality*, *necessity*, and *repeated reasoning*,² which will be discussed in the second paper. Collectively, the two papers present the core themes of *DNR*. A complete portrait of these themes will not come into full view until the end of the second paper. It may be useful, therefore, to begin with an initial image of this portrait through several standpoints, even if—unavoidably—they are expressed in terms of concepts that are yet to be precisely defined and include assertions that are yet to be justified. In a nutshell, these standpoints are:

1. Mathematics teaching must not appeal to gimmicks, entertainment, or contingencies of reward and punishment, but focus primarily on the learner’s *intellectual need* by fully utilizing humans’ remarkable capacity to be puzzled. Nor should mathematics curricula compromise the mathematical integrity of their contents. A curriculum is mathematical only if it adheres to and maintains the essential nature of the mathematics discipline. A “geometry curriculum,” for example, is not geometry if deductive reasoning is not among its *eventual* objectives.³ Teaching correct mathematics, however, is not necessarily correct teaching. A teacher may maintain the mathematical integrity of the content he or she is presenting but neglect the *intellectual need* of the students or be mistaken as to what constitutes such a need for them. As a proof-free “geometry curriculum” does not teach geometry, an intellectually-purposeless “algebra curriculum”—one in which students’

² Throughout the two papers, a term with a special *DNR* meaning is italicized until after it is first defined.

³ The adjective “eventual” is to acknowledge that deductive reasoning is developmental. The nature and level of rigor of deduction taught must carefully aligned with the current conceptual development of the students.

actions are socially rather than intellectually driven—does not teach students. In *DNR*-based instruction, the integrity of the content taught and the *intellectual need* of the student are equally central.

2. The mathematical integrity of a curricular content is determined by the *ways of understanding* and *ways of thinking* that have evolved in many centuries of mathematical practice and continue to be the ground for advances. An important goal of research in mathematics education is, therefore, to identify these *ways of understanding* and *ways of thinking* and recognize, when possible, their development in learners and in the history of mathematics, and, accordingly, develop and implement mathematics curricula that position the mathematical integrity of the content taught and *intellectual need* of students at the center of the instructional effort.
3. Since reasoning deductively is the single most central *way of thinking* in mathematics, mathematical proof should be a central focus of mathematics curricula.
4. The notions of *ways of understanding* and *ways of thinking* have specific, technical meanings; they will be defined precisely in the first section of this paper. Until then, they can be thought of as two different categories of knowledge, the first refers to “subject matter,” such as particular definitions, theorems, proofs, problems and their solutions, and the second to “conceptual tools,” such as heuristics and beliefs about what constitutes proof in mathematics.⁴ According to *DNR*, while knowledge of and focus on subject matter is indispensable for quality teaching, it is not sufficient. Mathematics instruction should also concentrate on conceptual tools.
5. The notion of *intellectual need* is inextricably linked to another notion: *epistemological justification*. The latter refers to the learner’s discernment of how and why a particular piece of knowledge came to be. It involves the learner’s perceived cause for the birth of knowledge. That perceived cause is a problematic situation whose resolution for the learner has necessitated for her or him the creation of a new knowledge. Such a situation is called *intellectual need*.
6. According to *DNR*, the central objectives of any mathematics curriculum must be formulated in terms of both *ways of understanding* and *ways of thinking*. Instructional principles to achieve these objectives should be based on (a) the developmental interdependency between these two categories of knowledge, (b) students’ intellectual needs, and (c) factors that facilitate internalization and organization of knowledge. These three elements correspond, respectively, to the instructional principles of *duality*, *necessity*, and *repeated reasoning*.

As was mentioned earlier, this paper addresses the question concerning the mathematics that should be taught in school, with a particular reference to proving. The paper is organized into four sections:

⁴ There are different reasons for using the terms *way of understanding* and *way of thinking*, rather than the terms, *subject matter* and *conceptual tool* (see Harel, 2008).

Section 1 offers a definition of mathematics in terms of *ways of understanding* and *ways of thinking*. As will be shown, these constructs are generalizations of the notions, *proof* and *proof scheme*, respectively. The latter have been discussed in various places in the literature.

Section 2 offers a classification of *ways of thinking* into *proof schemes*, *problem-solving approaches*, and *beliefs about mathematics*. Particular attention will be given to two *ways of thinking* within the *proof scheme* category: *result pattern generalization (RPG)* and *process pattern generalization (PPG)*—the former is an instance of the *empirical proof scheme*, whereas the latter of the *deductive proof scheme*.

Section 3 discusses curricular implications of the *DNR* definition of mathematics. The discussion will address three issues. The first issue concerns the consequence of this definition to the question: should *ways of understanding* and *ways of thinking* used or produced by individuals (students, for example) while they are engaged in a mathematical activity be considered mathematical even if they are narrow or faulty? The second issue concerns the lack of attention to *ways of thinking* by mathematicians and its curricular consequences. This concern will be discussed in the context of the development of *definitional reasoning*—the *way of thinking* by which one defines objects and prove assertions in terms of mathematical definitions. The third issue concerns the lack of attention to *ways of thinking* by teacher educators and its impact on their view of the mathematics teachers ought to know. This concern will be discussed in the context of Duval’s (1991) distinction between argumentation and proof.

Section 4 discusses methodological implications entailed from the constructs of *way of understanding* and *way of thinking*.

Section 5 concludes with a recap of the concepts and themes discussed in the paper.

1 Ways of Understanding and Ways of Thinking as Constituent Elements of Mathematics Curricula

In Harel (2008), I argued that from pedagogical and epistemological viewpoints, mathematics should be conceived of as a discipline consisting of two categories of knowledge: “subject matter” and “conceptual tools.” There I provided a precise characterization of these two categories of knowledge—calling the first *ways of understanding*, and the second *ways of thinking*—and offered a definition of mathematics in terms of these two constructs. To make this paper better self-contained, I will re-introduce these concepts, albeit briefly and in a new way that better explains their origin in my earlier work on proof. As we will see shortly, these two concepts are generalizations of the notions of *proof* and *proof scheme*, respectively. The latter have been discussed broadly in various places in the literature (see below).

1.1 Proving, Proof, and Proof Scheme

In Harel and Sowder (1998; 2007), proving is defined as the *mental act*⁵ a person (or community)⁶ employs to remove doubts about the truth of an assertion. The proving act is instantiated by one of two acts, *ascertaining* and *persuading*, or by a combination thereof. *Ascertaining* is the act an individual employs to remove her or his own doubts

⁵ In DNR, *mental act* is a primary term (see Harel, 2008)

⁶ In most contexts throughout the two papers, a reference to “individual” is applicable to “community; hence, it is not necessary from now on to mention both.

about the truth of an assertion, whereas *persuading* is the act an individual employs to remove others' doubts about the truth of an assertion. The following definitions are oriented within this subjective perspective:

A *proof* is the particular argument one produces to ascertain for oneself or to convince others that an assertion is true, whereas a *proof scheme* is a collective cognitive characteristic of the proofs one produces.

Consider the following example: When asked why 2 is an upper bound for the sequence, $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, ..., some undergraduate students produced the proof:

Proof A:

$\sqrt{2} = 1.41$, $\sqrt{2+\sqrt{2}} = 1.84$, $\sqrt{2+\sqrt{2+\sqrt{2}}} = 1.96$ [five more items of the sequence were evaluated] we see that [the results] are always less than 2, ... Hence, all items of the sequence are less than 2.

Other students produced the proof:

Proof B:

“Clearly, $\sqrt{2}$ is less than 2. The second item is less than 2 because it is the square root of a number that is smaller than 4, this number being the sum of 2 and a number that is smaller than 2. The same relationship exists between any two consecutive terms in the sequence.”

These two proofs (reported in Harel, 2001) are products resulting from carrying out the proving act, either in the form of ascertainment or persuasion. They may suggest certain persistent characteristics of these students' act of proving. For example, on the basis of additional observations of proofs produced by these two groups of students, we may characterize the proving act of the first group as empirical and that of the second group as deductive, if the respective proofs they produce are similar in nature to the ones presented here.

Thus, we have here a triad of concepts: proving act, proof, and proof scheme. A proof is a cognitive product of the proving act, and proof scheme is a cognitive characteristic of that act. Such a characteristic is a common property among one's proofs (see Figure 1).⁷

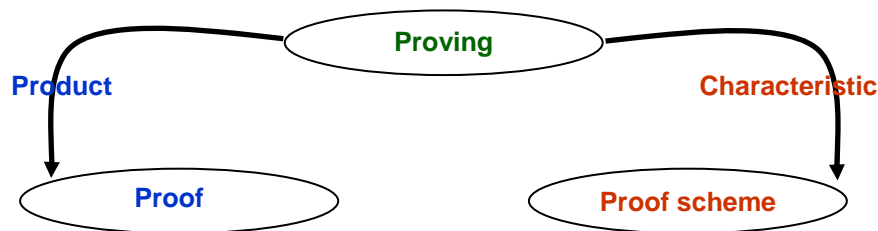


Figure 1: The triad of proving, proof, and proof scheme: a proof scheme is a common characteristic of proofs—the products of one's mental act of proving.

⁷ The question concerning the extent of the empirical observations of someone's proof that determines her or his proof scheme is a methodological question and will not be addressed in this series of papers.

Based on students' work and historical development, Harel and Sowder (1998) offered a taxonomy of proof schemes consisting of three classes: External Conviction, Empirical, and Deductive. Central elements of this taxonomy will be outlined in Section 1.3.1. For the complete taxonomy, see Harel and Sowder (1998) and Harel (2007) and for relations of the taxonomy to other taxonomies and to the functions of mathematical proof, see Harel and Sowder (2007).

1.2 A Definition of Mathematics

As I engaged deeply in the investigation of students' conceptions of proof, I came to realize that while the triad, proving, proof, and proof scheme, is useful, even critical, to understanding the processes of learning and teaching mathematical proof, it is insufficient to document and communicate clinical and classroom observations. This is so because proving itself is never carried out in isolation from other mental acts, such as "interpreting," "connecting," "modeling," "generalizing," "symbolizing," etc. As with the act of proving, we often wish to talk about the products and characteristics of such acts. Thus, the following definitions:

A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's *way of understanding* associated with that mental act. Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a *way of thinking* associated with that mental act.

Figure 2 depicts the triad of constructs, mental act, way of understanding, and way of thinking; hence, it is viewed as a generalization of Figure 1. This triad is part of the organizing constructs of *DNR*.

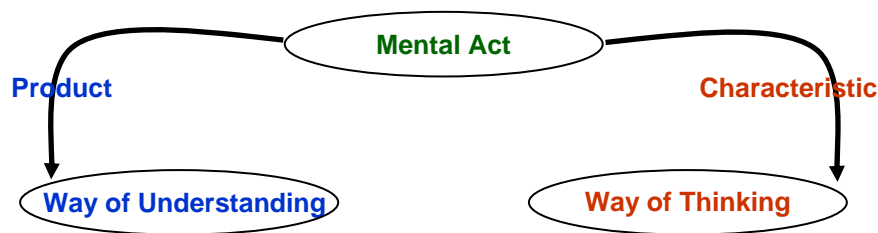


Figure 2: The triad, mental act, way of understanding, and way of thinking—a generalization of the triad, proving, proof, and proof scheme, respectively.

It is clear from these definitions that a proof is a way of understanding, whereas a proof scheme is a way of thinking. Likewise, in relation to the mental act of interpreting, for example, a particular interpretation one gives to a term, a statement, or a string of symbols is a way of understanding, whereas a cognitive characteristic of one's interpretations is a way of thinking associated with the interpreting act. For example, one's ways of understanding the string $y = -3x + 5$ may be: (a) an equation—a constraint on the quantities, x and y ; (b) a number-valued function—for an input x , there corresponds the output $y = -3x + 5$; (c) a truth-valued function—for an input (x, y) , there corresponds the output "True" or "False"; and (d) "a thing where what you do on the left you do on the right." The first three ways of understanding suggest a mature way

of thinking: that “symbols in mathematics represent quantities and quantitative relationships.” On the other hand, the fourth way of understanding, which was provided by a college freshman, is likely to suggest a *non-referential symbolic* way of thinking—a way of thinking where mathematical symbols are free of coherent quantitative or relational meaning. Other examples of ways of understanding and ways of thinking will emerge as the two papers unfold.

As the title of this section suggests, ways of understanding and ways of thinking are the constituent elements of mathematics. Mathematicians, the practitioners of the discipline of mathematics, practice mathematics by carrying out mental acts with particular characteristics—ways of thinking—to produce particular constructs—ways of understanding. Accordingly, in *DNR*, mathematics is defined as a discipline consisting of these two sets of knowledge. Specifically:

Mathematics is a union of two sets: The first set is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the *institutionalized*⁸ ways of understanding in mathematics throughout history. The second set consists of all the ways of thinking that are characteristics of the mental acts whose products comprise the first set. (Figure 3)

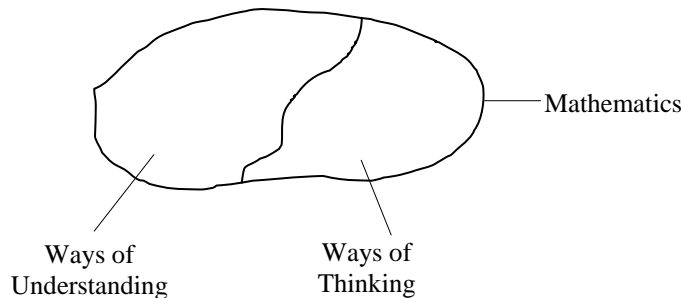


Figure 3: Mathematics is the union of two sets of knowledge: ways of understanding and ways of thinking. Denoting these three sets by M , WoU , and WoT , respectively, we have: $M = WoU \cup WoT$

The main pedagogical implication of this definition is that mathematics curricula at all grade levels, including curricula for teachers, should be thought of in terms of the constituent elements of mathematics—ways of understanding and ways of thinking—not only in terms of the former, as currently is largely the case. This point will be elaborated upon in Section 3. The triad, “mental act, way of understanding, way of thinking,” has, in addition, methodological implications; they will be discussed in Section 4.

2 Categories of Ways of Thinking

To further illustrate this triad of constructs, I will present in this section a crude classification of ways of thinking. This may help better understand the pedagogical and

⁸ *Institutionalized* ways of understanding are those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems. A subject matter of particular field may be viewed as a structure of institutionalized ways of understanding.

methodological implications that follow. A more refined classification is not reported in this paper due to space limitation. The classification emerged as we engaged in pedagogical and research activities of identifying, generating, and organizing ways of thinking. As will be explained in Section 4, methodologically, when analyzing a way of thinking employed by a student, it is crucial to specify the mental act associated with it. This classification provides an initial reference system for what that mental act might be.

Three, interrelated categories of ways of thinking in this classification are: *proof schemes*, *problem-solving approaches*, and *beliefs about mathematics*. The literature in mathematics education addresses each of these categories extensively. Here, however, the goal is merely to briefly describe each, with more elaboration on the first.

2.1 Proof Schemes

Based on students' work and historical development, Harel and Sowder (1998) offered a taxonomy of proof schemes consisting of three classes: *External Conviction*, *Empirical*, and *Deductive*. A brief description of essential elements of this taxonomy follows. This partial taxonomy is depicted in Table 1 (For a discussion of the complete taxonomy, see Harel, 2007; Harel & Sowder, 1998, 2007).

Proving within the *external conviction proof scheme* class depends on (a) an authority such as a teacher or a book (the *authoritative proof scheme*), (b) strictly the appearance of the argument (the *ritual proof scheme*, e.g., proofs in geometry must have a two-column format), or (c) symbol manipulations, with the symbols or the manipulations having no coherent system of referents in the eyes of the student (the *non-referential symbolic proof scheme*).

The *empirical proof scheme* class is marked by reliance on either (a) evidence from examples of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth (the *inductive proof scheme*), or (b) visual perceptions (the *perceptual proof scheme*).

The *deductive proof scheme* class consists of two categories: the *transformational proof scheme* category and the *modern axiomatic proof scheme* category. The essential characteristics of the former are: generality, operational thought, and logical inference. The generality characteristic has to do with an individual's understanding that the goal is to justify a "for all" argument, not isolated cases, and that no exception is accepted. The operational thought characteristic manifests itself when an individual forms goals and subgoals and attempts to anticipate their outcomes during the proving process. Finally, when an individual understands that justifying in mathematics must ultimately be based on logical inference rules, the logical inference characteristic is being employed. The *modern axiomatic proof scheme* category has all these characteristics but also includes the attribute that one understands that in principle any proving process must start from undefined terms and axioms and that these axioms determines one's mathematical reality. This is fundamentally different from Euclid's Elements, where humans' idealized special reality determines the axioms (see also Harel, 1999).

Table 1
A Partial Taxonomy of Proof Schemes

External Conviction	Empirical	Deductive
Authoritative	Inductive	Transformational

2.1.1 Result Pattern Generalization (RPG) vs Process Pattern Generalization (PPG)

Empirical reasoning is a common and robust way of thinking among many students and even among teachers. In what follows, I will discuss a special case of this way of thinking, called *result pattern generalization (RPG)*, together with its counterpart, *process pattern generalization (PPG)*, which belongs to the deductive proof schemes category. Proving that the general term of the sequence 1, 2, 4, 8, 16, ..., is 2^n because this is consistent with the given numerical values is *RPG*; proving the same fact by demonstrating that the process which generated this sequence is equivalent to repeated multiplication by two is *PPG*. Thus, process pattern generalization is a way of thinking in which one's proving is based on regularity in the process, though it might be initiated by regularity in the result, whereas result pattern generalization is a way of thinking in which one's proving is based solely on regularity in the result—obtained by substitution of numbers, for instance.

Proof A and Proof B discussed in Section 1.1 are manifestations of the *RPG* and *PPG* ways of thinking, respectively. The *RPG* way of thinking is prevalent among students and teachers. At least, this was the case among the teachers in my teaching experiments. Even in simple situations, where the application of *PPG* seems unproblematic, teachers evoked *RPG* spontaneously, as is illustrated by the following episode:

A group of algebra teachers participating in a summer algebra institute worked on a problem about square quilts made out of a grid of small congruent squares.⁹ The solution to the problem requires, among other things, expressing in general terms the number of squares on the main diagonals of such a quilt. The focus of the discussion here is only on this aspect of the solution; the full solution episode, while interesting, is not directly relevant to the *RPG-PPG* distinction. After some work, the teacher participants realized, by looking at a few special cases, that the expression sought depends on whether the number of squares on the side of the quilt is odd or even. In each of these cases, they obtained a correct expression by generalizing a particular pattern. In the even case, for example, they drew the first few quilts (e.g., 2-by-2, 4-by-4, and 6-by-6 quilts), counted

⁹ A company produces quilts made out of small congruent squares, where the squares on the main diagonals of the quilt are black and the rest are white. The quilts are square; that is, the length and width of the quilt are equal. The cost of a quilt is: for material, \$1 for a black square and \$0.50 for a white square; for labor, \$0.25 per square. To order a quilt, one must specify the number of black squares, the number of white squares, or the total number of squares. April, Bonnie, and Chad ordered three identical quilts, and each filled out an order form like the one below. April entered the number of black squares in the Black Cell. Bonnie and Chad entered the same number as April's, but accidentally Bonnie entered her number in the Whites Cell and Chad in the Total Cell. April was charged \$139.25. How much were Bonnie and Chad charged?

Number of Black Squares	Number of White Squares	Total of Squares

the total number of squares on the main diagonals in each case and recorded the results: 4, 8, and 12. Upon inspection, they observed that each of the three numbers is a multiple of 4. They then generalized this particular pattern to assert that the total number of squares on the main diagonals is $4n$ for any even quilt of size $2n$ —a clear application of RPG.

A few teachers provided PPG-based proofs. Tanya¹⁰ was one of these teachers. Initially, Tanya arrived at the expression $4n$ through RPG way of thinking like her classmates. Then she considered a few quilts (2 by 2, 4 by 4, and 6 by 6 quilts) and showed for each one that each square on a main diagonal corresponds to one and only one square on the side of the quilt, and vice versa. She continued by making statements to the effect that this property is invariant under the quilt's size, and so the one-to-one correspondence she pointed to in the special cases holds for any quilt. As she defended this claim, she drew and kept pointing to a sketch of a quilt of an unspecified size. In this response, Tanya seems to be generalizing a process, that of constructing a one-to-one correspondence between the squares on a main diagonal and the squares on the side of the quilt, observing that she could, in principle, carry out that process for any given even-size quilt. Thus, Tanya's solution manifests an instance of a PPG way of thinking.

The question of how to help learners transition from RPG to PPG has been the focus of my research for some time, and it belongs to the discussion of the second question addressed by this series of publications. An extensive research report detailing the instructional implementation of *DNR* to achieve this goal is underway (Harel, Fuller, Manaster, & Soto, in preparation).

2.2 Problem Solving Approaches

Problem-solving approaches are ways of thinking associated with the problem-solving act. While an actual solution (correct or erroneous) one provides to a problem is a way of understanding—because it is a cognitive product of the person's problem-solving act—a problem-solving approach is a cognitive characteristic of that act and hence is a way of thinking. For example, problem-solving approaches such as “look for a simpler problem,” “consider alternative possibilities while attempting to solve a problem,” and “just look for key words in the problem statement” characterize, at least partially, the problem-solving act; hence, they are instances of ways of thinking. Any heuristic is a problem-solving approach, and, hence, heuristics are ways of thinking. In the literature, the term heuristic is often used for successful problem-solving approaches: “Heuristic strategies are rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution” (Schoenfeld, 1985, p. 23). I use the term “heuristic” in this sense. The judgment as to whether a problem-solving approach is a heuristic is from the viewpoint of the observer, not the observed. Observers who are mathematically experienced individuals are unlikely to judge the approach “look for a key word in the problem statement” as a heuristic. Students, on the other hand, may deem this approach successful if they have applied it “successfully” in a large number of cases—a common scenario in current school mathematics due the types of problems students are assigned.

¹⁰ Pseudonym

2.3 Beliefs about Mathematics

Beliefs about mathematics are one's views of mathematics itself. Specifically, they are the characteristics of one's interpretation of (a) what mathematics is or is not, (b) how it is created, and (c) its intellectual or practical benefits. Examples of these beliefs include, respectively, (a) "Formal mathematics has little or nothing to do with real thinking or problem solving" (Schoenfeld, 1985), "Justification that does not include algebraic symbols is not a proof" (Harel & Sowder, 1998) (b) "The solution of a problem should not take more than five minutes" (Schoenfeld, 1985), "Proofs are created quickly because that is how my professor create them on the board" (Harel & Sowder, 1998), and (c) "It is advantageous to have multiple interpretations of a mathematical concept" (Harel, 1998).

Of particular relevance to this paper is the belief that the benefit of proving goes beyond establishing truth, a thesis articulated by Rav (1999), who convincingly argues "... conceptual and methodological innovations are inextricably bound to the search for and the discovery of proofs, thereby establishing links between theories, systematizing knowledge, and spurring further developments."

Figure 4 depicts the classification of ways of thinking described above. This figure, thus, is an extension of Figure 2. Although each of the proof schemes categories, *external conviction*, *empirical*, and *deductive*, is further classified, in Figure 4 this classification is not shown in order to avoid overburden.

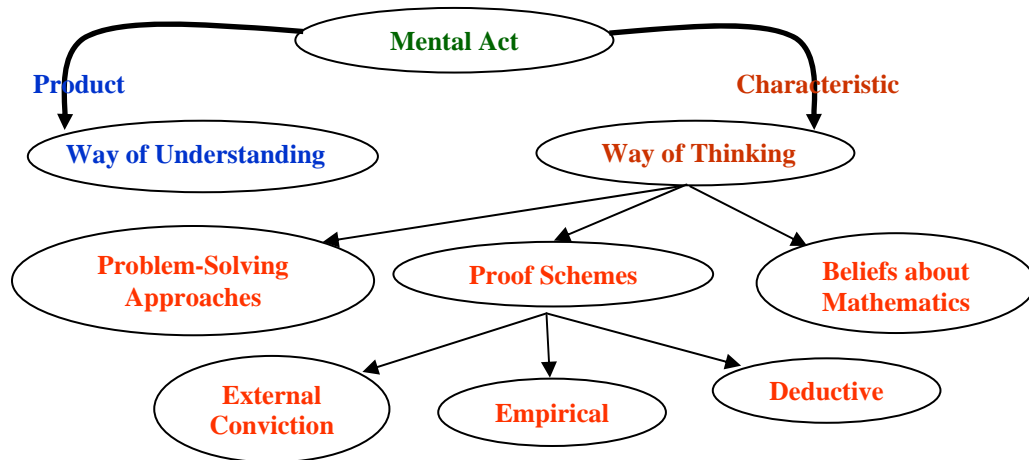


Figure 4: The triad, mental act, way of understanding, and way of thinking; the three categories, problem-solving approaches, proof schemes, and beliefs about mathematics, comprising ways of thinking; and the three categories, external conviction, empirical, and deductive, comprising proof schemes.

3 Curricular Implications

In essence, the above *DNR* definition of mathematics provides a core answer to the first of the questions this series of papers addresses, namely: What is the mathematics that should be taught in school? Specifically, implied from this definition is that mathematics curricula at all grade levels, including curricula for teachers, should be thought of in terms of the constituent elements of mathematics, ways of understanding and ways of thinking, not in only in terms of the former, as currently is largely the case. Instructional objectives concerning ways of thinking should be formulated in terms of

problem-solving approaches, proof schemes, and beliefs about mathematics. In Harel (2008), I illustrated this claim in the context of curricular materials in arithmetic, algebra, and proof. Here, I will discuss this claim in three additional contexts:

1. Should ways of understanding and ways of thinking used or produced by individuals (students, for example) while they are engaged in a mathematical activity be considered mathematical even if they are narrow or faulty? This question leads to a distinction between *desirable* and *institutionalized* mathematics and, accordingly, to a crucial instructional implication.
2. Lack of attention to ways of thinking by mathematicians: In this discussion, I focus mainly on the mathematicians' view of sequencing curricular content according to a logical structure of that content, paying no or scant attention to the development of *definitional reasoning*, a crucial way of thinking that is inextricably linked to the development of the deductive proof scheme.
3. Lack of attention to ways of thinking by teacher educators: In this discussion, I claim that with the perception of mathematics as a discipline consisting of ways of understanding, it is difficult to form a pedagogically convincing position on the question: What mathematics do mathematics teachers need to know to be effective? I will illustrate my claim with a discussion about argumentation versus proof.

3.1 Desirable versus Institutionalized Mathematics

A consequence of the *DNR* definition of mathematics is that mathematics must include ways of understanding and ways of thinking that from the vantage point of contemporary mathematicians are imperfect or erroneous; Euclid's *Elements* is an example of a correct, yet imperfect, piece of mathematics. This leads to the following question: Should ways of understanding and ways of thinking used or produced by individuals (students, for example) while they are engaged in a mathematical activity be considered mathematical even if they are narrow or faulty? *DNR*'s answer to this question is affirmative in so far as the individual has utilized—with or without the help of an expert—such ways of understanding and ways of thinking for the construction of institutionalized knowledge—knowledge accepted by the mathematics community at large.

This position is consistent with the definitions of way of understanding and way of thinking. As can be seen from the examples discussed in the previous section, these terms do not imply correct knowledge. The terms only indicate the knowledge currently held by a person, which may be correct or erroneous, useful or impractical. Having said this, it must be emphasized that in *DNR* the ultimate goal of instruction must be unambiguous: to help students develop ways of understanding and ways of thinking that are compatible with those currently accepted by the mathematics community at large. In *DNR*, this goal is meaningless without considering one of *DNR*'s premises, which, after Piaget, asserts that the process of knowing proceeds through a continual tension between accommodation and assimilation, and, thus, implies that the process of learning necessarily involves the construction of imperfect and even erroneous ways of understanding and deficient, or even faulty, ways of thinking. Teachers must be aware of this natural phenomenon when working toward a cognitive goal, and their teaching behaviors must be consonant with it. For the implications of this premise, see Thompson (1985). In particular, they must continually build models for students' current ways of

understanding and ways of thinking—what Steffe and Thompson (2000) call “mathematics of students”—regardless of their quality, and help students gradually refine and modify them toward those that have been institutionalized—those the mathematics community at large accepts as correct and useful in solving mathematical problems. *Desirable* ways of understanding and ways of thinking are those a teacher sets as intermediate cognitive objectives toward those held and practiced by the mathematics community. Hence, any institutionalized way of understanding or ways of thinking is also desirable (see Figure 5).

To better illustrate the distinction between “desirable” and “institutionalized,” consider the following example which concerns the concept of “mathematical induction.” Several researchers (e.g. Dubinsky, 1986, 1989; Ernest, 1984) have shown that students have major difficulties with this concept. One of my teaching experiments (Harel, 2001) aimed specifically at this important concept. There are several reasons for the difficulties students experience with mathematical induction; the one that is relevant to the discussion here is that the standard instructional treatments introduce the formal principle of mathematical induction too abruptly. The new instructional treatment was designed to help students develop the principle of mathematical induction gradually. It consisted of four instructional phases, where only in the fourth phase could the students fully internalize this principle. The first three phases had intermediate cognitive objectives consisting mostly of desirable, not institutionalized, ways of thinking. For example, in Phase 1, the students were engaged for a relatively long period of time in working on problems typified by:¹¹

- Find an upper bound to the sequence $\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \dots$
- Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces, where each piece covers three squares.

The general proofs that emerged from these problems were PPG-based, as was exemplified by Proof A in Section 1.1. Thus, the intermediate, desirable cognitive objective of this phase was for students to attend to the relationship between the arguments on neighboring natural numbers, n and $n + 1$, not to prove the proposition by the institutionalized approach, that is, the formal principle of mathematical induction. From a purely mathematical viewpoint, these problems involve recursively defined functions and, hence, their proofs must involve mathematical induction. However, this view requires the axiomatic proof scheme, which was beyond the capability range of these students at this stage in their conceptual development. Initially, however, not all of students’ responses had a PPG characteristic—many were empirical; Proof B in Section 1.1 is an example.

This instructional treatment was successful in that by the end of the fourth phase, a decisive majority of the students in the teaching experiment, including those who first used empirical justification such as this, internalized the formal principle of mathematical induction as a proof scheme, a method by which they ascertain for themselves and persuade others about the truth of mathematical assertions on the natural numbers.

¹¹ For the exact characteristics of these and other induction problems, see Harel (2001)

This discussion points also to an avoidable tension in mathematics curriculum and instruction between desirable and institutionalized mathematical knowledge.¹² Other scholars have recognized this tension. It is a significant element in Brousseau's (1997) theory of *Didactique*, and Movshovitz-Hadar (1993) discusses its manifestation in a particular problem related to mathematical induction.

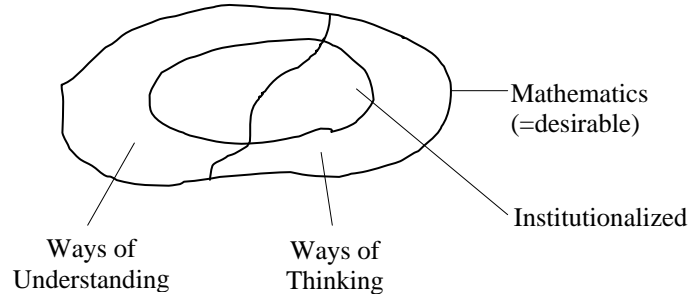


Figure 5: Denoting by I and D , respectively, the sets of institutionalized and desirable ways of understanding and ways of thinking, and building on Figure 3, we have the following relations: $M = WoU \cup WoT$, $I \subset M$, and $M = D$.

3.2 Mathematicians' Perception of Mathematics: A Particular Reference to Definitional Reasoning

The DNR definition of mathematics is consistent with mathematicians' practice of mathematics but not with their perception of it. There is a fundamental difference between the way mathematicians perceive mathematics and the way they practice it in their research. Mathematicians' perception of mathematics manifests itself in how they communicate it in formal professional forums, such as conference presentations and journal publications, and in educational forums, when they teach it or write textbooks on its topics. The main, explicit focus in both forums is on ways of understanding—the content of axioms, definitions, theorems, proofs, etc., and their logical structure. Mathematicians' research practice, which includes learning and discovering such contents, utilizes numerous ways of thinking, but these usually are not part of their perception of mathematics. This is not surprising. In order for one's practice of a discipline to be part of one's conscious image of that discipline, one must be aware of how one comes to know and discover things in the discipline. However, attention to and reflection on the processes of knowing and discovering for the purpose of creating conditions that others come to know and discover are not integral parts of the mathematician's training.

Some mathematicians see a need to defend the thesis put forth in this paper; namely, that both ways of understanding and ways of thinking are constituent elements of mathematics:

The main thesis to be developed [in this paper] is that the essence of mathematics resides in inventing methods, tools, strategies [i.e., ways of thinking] and concepts [i.g., ways of understanding] for solving problems which happen to be on the

¹² Given the definition of *mathematics* presented earlier, the term *mathematical knowledge* refers collectively, here and elsewhere in the two papers, to desirable and institutionalized ways of understanding and ways of thinking.

current internal research agenda or suggested by some external application.” (Rav, 1999, p. 6)

The incomplete perception of mathematics discussed above is one source of a simplistic conceptualization of mathematics learning and of teaching. Viewing mathematics merely in terms of ways of understanding reduces curriculum development to a matter of sequencing curricular content according to a logical structure of that content, paying no or scant attention to critical ways of thinking and to the complexity of the process involved in acquiring and internalizing them. The effective trajectory for learning—and therefore for teaching—a particular mathematical topic, according to this perception, is one that parallels a logical structure of that topic. How should, for example, the concept of fraction be introduced in the fifth grade? The answer according to this conceptualization is simple: start with assumptions (i.e., axioms) and mathematical definitions and proceed through logical deduction to build the curricular content related to this concept (see, for example, Wu, 2007). This view, however, is irreconcilable with research findings, which reveal a different and more complex terrain for the learning of fractions (Kieren, 1992). More generally, reasoning in terms of mathematical definition and justifying assertions deductively are ways of thinking not available to fifth graders. This knowledge is developmental and does not become an integral part of the repertoire of students’ ways of thinking until advanced grades (if at all).

Students’ definitions of a concept and the way the concept is defined in a curriculum may or may not be mathematical. Generally speaking, a definition is mathematical if it is a description that applies to all objects to be defined and only to them. Typically, students’ definitions of concepts are not mathematical, even if the concepts were defined to them mathematically. Understanding the notion of mathematical definition and appreciating the role and value of mathematical definitions in proving is a developmental process, which is not achieved for most students until adulthood. I refer to the way of thinking by which one defines objects and proves assertions in terms of mathematical definitions as *definitional reasoning*. A crucial feature of this way of thinking, which belongs to the transformational proof schemes category, if not to the modern axiomatic proof schemes category, is that with it one is compelled to conclude logically that there can be only one mathematical definition for a concept within a given theory; namely, if D_1 and D_2 are such definitions for a concept C , then D_1 is a logical consequence of D_2 , or vice versa; otherwise, C is not well defined.

Many students even in advanced grade do not possess definitional reasoning. For example, in Van Hiele’s (1980) model only in the highest stage of geometric reasoning are students’ definitions of Euclidean objects mathematical (see Burger & Shaughnessy, 1986). Definitional reasoning is largely absent among college students as well, even among undergraduate mathematics and engineering majors (Harel, 1999). For example, asked to define “invertible matrix,” many linear algebra students stated a series of equivalent properties (e.g., “a square matrix with a non-zero determinant,” “a square matrix with full rank,” etc.) rather than a definition. The fact that they provided more than one such property is an indication that they were not thinking in terms of mathematical definition.

Undoubtedly, definitional reasoning must be a target for mathematics instruction, but it cannot be assumed—definitely not among fifth-graders. The lack of recognition by teachers, researchers, and curriculum developers of these and other ways of thinking and

the nature of their development with learners is, I believe, one of the reasons for the failure of instruction to help students acquire them.

The question of how instruction can promote ways of thinking such as definitional reasoning is crucial. It belongs to the second question mentioned in the opening of this paper: How should mathematics be taught? As was explained earlier, this question will be discussed in the second paper.

3.3 School Teachers' Knowledge¹³: A Particular Reference to Argumentation versus Proof

School teachers' perception of mathematics tends to be consistent with that of their masters, the mathematicians, in that they, too, view the mathematics they teach mainly in terms of subject matter—though often their understanding of it is limited.

With the perception of mathematics as a discipline consisting of ways of understanding, it is difficult to form a pedagogically convincing position on the question, what mathematics do mathematics teachers need to know to be effective? Everyone agrees that teachers need to know what they teach, and they need to know more so that they have a sense where students are heading (Ball & Bass, 2000). But what and how much more mathematics do, say, second-grade teachers need to know? Do they need to know algebra, geometry, and probability; if so, at what level? Is introductory high-school algebra sufficient? Why not advanced high-school algebra? Maybe linear algebra is also needed? Do they need to master geometry proofs? Any position on these questions in terms of subject matter, i.e., ways of understanding, leads to a quandary: Why does a second-grade teacher need to know anything beyond basic arithmetic and properties of basic geometric figures?

Thinking of teachers' mathematical knowledge in terms of ways of thinking helps address this question more decisively. As we have discussed earlier, implied from the *DNR* definition of mathematics is that the role of the teacher is to advance students' ways of thinking, not only ways of understanding. Hence, an essential question to address is: what ways of understanding and ways of thinking should be targeted by second-grade mathematics curricula? This, in turn, necessitates the question: what ways of understanding and ways of thinking must a second-grade teacher possess? As an example, consider this question in relation to the proving act—the focus of this paper.

A crucial way of thinking that second graders should be exposed to, and, hence, their teachers must master involves the ability to reason in terms of underlying structures of numerical and geometric patterns; that is, in terms of PPG. Elementary arithmetic, suitable for second graders, can offer numerous opportunities to engage children in reasoning in PPG mode. However, if the teacher's dominant way of thinking is RPG rather than PPG—as is currently the case for many teachers—the empirical proof scheme, rather than the deductive proof scheme, would establish strong roots in the children's mathematical behavior. Research has shown that students—even early grade students—are capable of producing PPG- and transformational-based proofs (see, for example, Cobb et al., 1991; Maher & Martino, 1996). Obviously, for these ways of thinking to be fostered by teachers, they must be an integral part of the teachers' repertoire of reasoning.

¹³ The issue of teacher knowledge will be discussed in more detail in Paper II. Also in this paper, *teacher's knowledge base* will be defined in terms of *DNR* concepts.

In general, the question “What mathematical ways of thinking should a teacher at a particular grade level should possess?” is a research question. Its answer depends on various factors, including (a) the overall goals of mathematics curricula as are set by the mathematics education leadership, (b) constraints on knowledge development of concepts and operations (e.g., constraints on the nature of operators in multiplicative word problems, see Fischbein, Deri, Nello, & Marino, 1985), (c) models for learning trajectories (e.g., the action-process-object-schema model of Dubinsky, 1994), (d) understanding the nature and significance of teachers’ content knowledge (e.g. Ball & Bass, 2000), and (e) empirical findings on the effect of teachers’ knowledge of content on student learning (e.g. Hill, Rowan, & Ball, 2005). Considering the first factor in this list, I would argue that teachers at all grade levels must understand the structural nature of deductive reasoning. I explain:

The attention given to proof and argumentation in mathematics education research and in agenda documents such as the NCTM Principles and Standards for School Mathematics (NCTM, 2000) during the last decade attests to the significance the mathematics education community attributes to deductive reasoning. During this period, a major effort has been underway (at least in the USA) to change the current mathematics classroom climate by, among other things, promoting argumentation, debate, and persuasive discourse. There is no doubt that this is a worthwhile, and even essential, effort. However, there is a major gap between “argumentation” and “mathematical proof” that, if not understood by teachers, could lead them to advance mostly argumentation skills and little or no deductive reasoning. In mathematical deduction, one must distinguish between “status” and “content” of a proposition (see Duval, 2002). Status (e.g., hypothesis, conclusion, etc.), in contrast to content, is dependent only on the organization of deduction and organization of knowledge. Hence, the validity of a proposition in mathematics—unlike in any other field—can be determined only by its place in logical value, not by epistemic value (degree of trust). Students mostly focus on content and experience major difficulties detaching status from content. As a consequence, many propositions in mathematics seem trivial to students because they judge them in terms of epistemic values rather than logical values. For example, when a proposition $a \Rightarrow b$ is to be proven and the students view the statement b as self-evident, they are likely to experience difficulties with proofs that assume not b ; they are unable to separate the content of b from its status. In addition, in the process of constructing a proof, the status of propositions changes: the conclusion of one deductive step may become a hypothesis of another. These are crucial characteristics that must hold in any form of mathematical discourse, informal as well as formal (!). In argumentation and persuasion outside mathematics, on the other hand, they are not the main concern: the strength of the arguments that are put forward for or against a claim matters much more. Teachers, even second-grade teachers, must have appreciation for these differences; otherwise, they would be promoting argumentation rather than deductive reasoning, and their communication with their students would be unlikely to lay the seeds for this reasoning. Hence, prospective primary teachers must be introduced to ways of thinking such as definitional reasoning and deductive proof schemes.

4. Methodological Implications

Ways of understanding and ways of thinking are central in *DNR* both pedagogically and methodologically. We have already discussed in the preceding

sections their role as elements of mathematics curricula for students and teachers. In Paper II, we will further discuss the developmental interdependency between ways of understanding and ways of thinking. In what follows, I will conclude with a discussion of methodological issues concerning investigations of students' ways of understanding and ways of thinking.

The concepts of *way of understanding* and *way of thinking* are related to but different from the dichotomy between *product* and *process* commonly used in the mathematics education literature. Process often refers to a cognitive mechanism of how students retrieve, represent, utilize, or develop mathematical knowledge. Many theoretical models for such processes have been offered in the literature. Cobb (2007) classifies them into two categories: those that seek to account for students' reasoning in any mathematical domain (e.g. Dubinsky, 1991, APOS model; Pirie & Kieren, 1994, recursive model; Vergnaud, 1982, theorem-in-action model), and those that are domain-specific (e.g., the early-number learning models by Carpenter & Moser, 1984; Fuson, 1992; Steffe, Cobb, & Glasersfeld, 1988). While "the focus of process analyses is on the means used to obtain a particular result; of product analyses, on the results obtained" (Schoenfeld, 1987): "Educational research has traditionally emphasized product, the bottom line being the correct answer to the number of problems a student, or a group of students, can correctly answer." (p. 8)

It is not all clear under this distinction whether, for example, a student's justification of an assertion, interpretation of a problem or string of symbols, or generalization of an observation, is a product or a process. In *DNR* these are all products—ways of understanding—albeit of different mental acts: proving, interpreting, and generalizing, respectively. Ways of understanding are indispensable both instructionally and methodologically. Instructionally, it is the formation of new ways of understanding or reformation of existing ones that brings about the construction of new ways of thinking or a change in existing ones. This is part of the *duality principle*, which will be discussed in length in Paper II. Methodologically, the triad, "mental act, way of understanding, and way of thinking," offers an operational structure for conceptual analyses, which can be part of any model for students' reasoning. This structure involves three essential activities:

- (a) Specifying the particular mental acts of interest (e.g., interpreting, proving, generalizing, etc.).
- (b) Identifying products of each mental act in a form of utterances and actions; these are the ways of understanding associated with the respective act.
- (c) Inferring common cognitive characteristics among multitude of products of each act; these are the ways of thinking associated with the respective act.

The specification and isolation of the mental acts under consideration is methodologically crucial. Consider a researcher (or teacher) who follows students' solutions to problems on the order relation between fractions. The researcher focuses on two students, Ashley and Becky, and observes that for a given pair of fractions a/b and c/d , Ashley typically answers by comparing the order relation between the differences $a - b$ and $c - d$, whereas Becky answers it by comparing the order relation between the quotients $a \div b$ and $c \div d$. Based on these observed ways of understanding, the researcher may conclude that while Ashley is an additive reasoner, Becky is a proportional reasoner. Is this a sound judgment? "Additive reasoning" and "proportional

reasoning” are indeed ways of thinking, for they characterize how one models relations between covarying quantities. The data used by the researcher, however, may not be sufficient to arrive at this conclusion. It is so because as described there is no evidence that Ashley and Becky carry out the modeling act. Their solutions might be a mere symbolic game: a prescribed or invented procedure that involves little or no quantitative considerations. In other words, the students might have been transforming symbols with no attention to quantitative referents, in which case they would be employing the non-referential symbolic way of thinking—a characteristic of the symbol manipulation act.

While it is essential to ensure that the ways of understanding observed are associated with the mental acts considered, it is equally crucial to recognize that mental acts are not independent of each other; rather, one act is likely to be carried out in the context of, or to facilitate, other acts. For example, one must interpret in order to justify, represent in order to model, define in order to classify, etc. Further, in certain situations mental acts are performed concurrently rather than sequentially, and so it is difficult to discuss one mental act without evoking others. Boero et al. (Boero, Garuti, Lemut, & Mariotti, 1996), for example, showed that in some cases students’ conjecturing and justifying occur concurrently; namely, as a student forms a hypothesis, he or she also proves it—a phenomenon related to a construct of “cognitive unity”: a continuity between the production of a conjecture and the possible construction of its proof (see Pedemonte, this volume).

As one analyzes a way of thinking employed by a student, the classification presented in Section 2 can provide an initial reference system for what the mental act associated with that way of thinking might be.

5. Conclusion

This is the first in a series of two papers whose goal is to contribute to the debate on a pair of questions: What is the mathematics that we should teach in school? How should we teach it? These are fundamental questions in the discussion of educational standards (e.g., *NCTM curriculum and evaluation standards*, 2000), the concept of mathematical competence (Niss, 2001), and the theoretical basis of international comparative studies like PISA (Harlen, 2001) and TIMSS (Mullis et al., 2000).¹⁴ This paper addresses the first question from the *DNR* theoretical framework. According to this framework, the constitutive elements of mathematics, and therefore of mathematics curricula, are ways of understanding and ways of thinking. The first element refers to products of a mental act, whereas the second refers to their cognitive characteristics. The triad mental act, way of understanding and way of thinking, is a generalization of the triad proving, proof, and proof scheme, respectively, which emerged in investigations concerning the learning and teaching of mathematical proof. This generalization was necessitated out of the realization that the processes of learning and teaching mathematical proof involve numerous mental acts, such as “interpreting,” “connecting,” “modeling,” “generalizing,” “searching,” and “symbolizing,” and so attention to proving alone is insufficient to identify and communicate classroom and clinical observations.

Three main, not mutually exclusive, categories of ways of thinking are discussed in this paper: problem-solving approaches, proof schemes, and beliefs about

¹⁴ I thank the anonymous reviewer who recommended to list these examples of critical issues in which the two fundamental questions must be addressed.

mathematics. The classification can help identifying, generating, and organizing ways of thinking, and orient an analysis as to the mental act associated with a way of thinking under consideration.

Based on students' work and historical development, Harel and Sowder (1998) offered a taxonomy of proof schemes consisting of three classes: External Conviction, Empirical, and Deductive. A special case of the empirical proof scheme is the RPG way of thinking, whereas its counterpart, the PPG way of thinking, which manifests one's ability to reason in terms of underlying structures of numerical and geometric patterns, is an instance of and precursor for the deductive proof scheme.

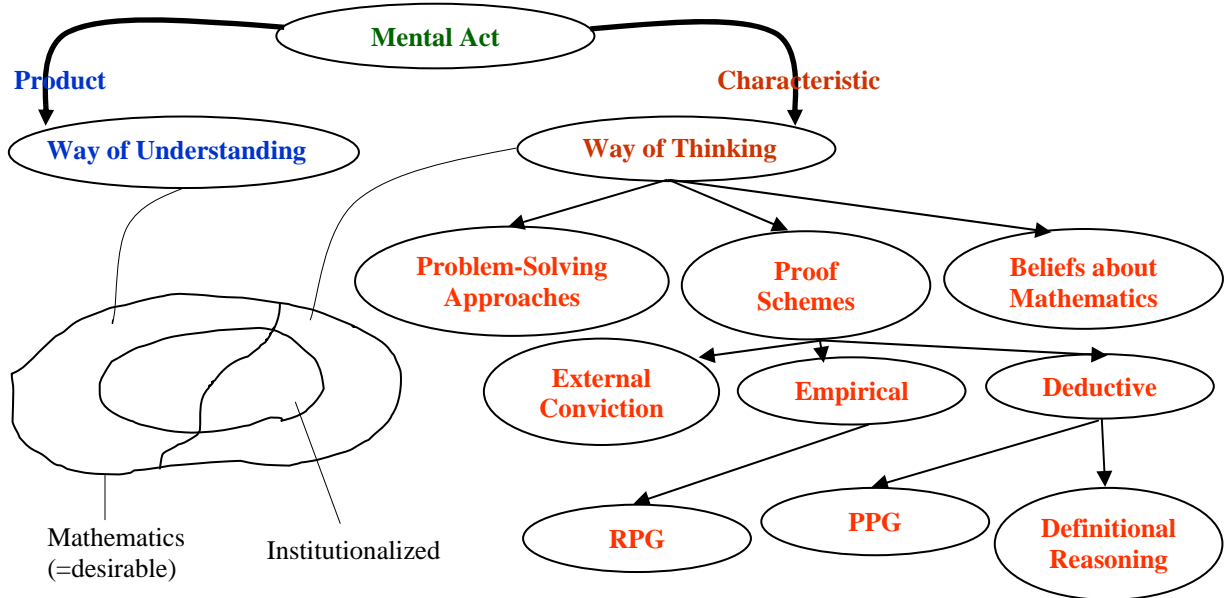
The *DNR* definition of mathematics entails several crucial curricular implications as to what mathematics to teach and why to teach it. One crucial implication is that mathematics curricula at all grade levels, including curricula for teachers, should be thought of in terms ways of understanding and ways of thinking, not only in terms of the former, as currently is largely the case. However, teachers at all grade levels—including college instructors—tend to view mathematics largely in terms of ways of understanding. While in *DNR* knowledge of and focus on subject matter is indispensable for quality teaching—a claim captured by the *duality principle*, which will be discussed in the subsequent paper—this knowledge is not sufficient. Teachers should also concentrate on ways of thinking.

The inclusion of all the ways of understanding and ways of thinking that have evolved throughout history in the definition of *mathematics* has a second crucial implication; namely, that teachers must continually build models for students' current ways of understanding and ways of thinking, regardless of their quality, and help students gradually refine and modify them toward desirable ones. This implication is based on a central premise of *DNR*; namely that, after Piaget, the process of knowing proceeds through a continual tension between accommodation and assimilation, and, thus, the process of learning necessarily involves the construction of imperfect and even erroneous ways of understanding and deficient, or even faulty, ways of thinking.

Viewing mathematics merely in terms of ways of understanding reduces curriculum development to a matter of sequencing curricular content according to a logical structure of that content, paying no or scant attention to critical ways of thinking and to the complexity of the process involved in acquiring and internalizing them. Definitional reasoning is a case in point. Thinking of teachers' mathematical knowledge in terms of ways of thinking, not only in terms of ways of understanding, on the other hand, compels one to address the question: What ways of thinking should be targeted by mathematics curricula for school mathematics? This, in turn, necessitates the question: What ways of thinking should be targeted by teacher education curricula? The PPG way of thinking, for example, is a crucial way of thinking that early grade students must be exposed to and, hence, their teachers must master, in order to be able to help students transition from empirical proof schemes to deductive proof schemes. Elementary arithmetic and basic geometry, suitable for early grade students, can offer numerous opportunities to engage these students in problems that can help them develop the PPG way of thinking. However, if the teacher's dominant way of thinking is RPG rather than PPG, as is often the case, the empirical proof scheme, rather than the deductive proof scheme, would establish strong roots in young students' mathematical behavior. Further, given the focus on proof and argumentation in current documents such

as the NCTM Principles and Standards for School Mathematics (NCTM, 2000), there is a need for teachers to understand the difference between “argumentation” and “mathematical proof;” without it, teachers would likely be advancing argumentation skills and little or no deductive reasoning.

Figure 6 depicts all the concepts depicted in the preceding five figures. They comprise the main concepts discussed in this paper.



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