Two Fundamental Questions: A *DNR* Perspective

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Introduction

The goal of my talk is to contribute to the debate on a pair of questions that are on the mind of many mathematics educators—teachers, teacher leaders, curriculum developers, and researchers who study the processes of learning and teaching—namely:

1. What is the mathematics that we should teach in school?
2. How should we teach it?

Clearly, one presentation is not sufficient to address these colossal questions, which are inextricably linked to other difficult questions—about student learning, teacher knowledge, school culture, societal need, and educational policy, to mention a few. My goal in this presentation is merely to articulate a pedagogical stance on these two questions. The stance is not limited to a particular mathematical area or grade level; rather, it encompasses the learning and teaching of mathematics in general.

This stance is oriented within a theoretical framework, called *DNR-based instruction in mathematics* (*DNR*, for short). The initials *D, N*, are *R* stand for three foundational instructional principles in the framework: *duality, necessity,* and *repeated reasoning*. *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts*—constructs defined and oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning.

The talk has five parts: The first part discusses *DNR* premises. The second and third parts of the talk deal, respectively, with the above two questions; namely, the second part focuses on the constituent elements of mathematics curricula and the third on the constituent elements of mathematics teaching. However, since one's approach to teaching necessarily depends on one's views of learning, the third parts devotes a portion of its space to the *DNR*'s definition of learning. The fourth, and final, part of the talk illustrates with an example of actual lesson how *DNR* is implemented in the classroom.

In what follow, I will give a synopsis of each of these four parts.

Part I: *DNR* Premises

DNR has eight premises. They are loosely organized in four categories:

1. **Mathematics**
    - ***Mathematics:*** Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history.
2. **Learning**
    - ***Epistemophilia:*** Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience.

- ***Knowing:*** Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.
- ***Knowing-Knowledge Linkage****:* Any piece of knowledge humans possess is an outcome of their resolution of a problematic situation.
- ***Context-Content Dependency:*** Learning is context and content dependent.

3. **Teaching**
   - ***Teaching:*** Learning scientific knowledge (such as mathematics) is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance.
4. **Ontology**
   - ***Subjectivity:*** Any observations humans claim to have made is due to what their mental structure attributes to their environment.
   - ***Interdependency****:* Humans' actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions.

Part II: **Constituent Elements of Mathematics Curriculum**

What is the mathematics that we should teach in school? *DNR'*s position on this question, based on its first premise, is that the constituent elements of mathematics, and therefore of desirable mathematics curricula, are ways of understanding *and* ways of thinking. A way of understanding is a product of a mental act, whereas a way of thinking is a characteristics of ways of understanding associated with such an act. The triad "mental act," "way of understanding," and "way of thinking," is central in *DNR.* It is a generalization of the triad "proving, proof, and proof scheme," which emerged in investigations concerning the learning and teaching of mathematical proof (see, for example, Harel & Sowder, 1998, 2007). In a nutshell, according to *DNR* the answer to the first question in the opening of this narrative should be driven by desirable ways of understanding *and* ways of thinking, not only by the former, as is currently the case. An important goal of research in mathematics education is, therefore, to identify these ways of understanding and ways of thinking*;* recognize, when possible, their development in the history of mathematics; and, accordingly, develop mathematics curricula and teacher education programs that aim at helping students construct them.

Part III: Constituent Elements of Mathematics Learning and Teaching

In *DNR*, the definition of mathematics learning follows from the *DNR* premises. It follows from the Learning-Knowledge Linkage Premise that problem solving is the means—the only means—to learn. When one encounters a problematic situation, one necessarily experiences phases of disequilibrium, often intermediated by phases of equilibrium. Disequilibrium, or perturbation, is a state reached when one encounters an obstacle. *DNR* defines perturbation in terms two types of human needs: *intellectual need* and *psychological need—*terms to be discussed in the next section:

> ***Learning*** is a continuum of disequilibrium-equilibrium phases manifested by (a) intellectual *and* psychological needs that instigate *or* result from these phases and (a) ways of understanding *or* ways of thinking that are utilized *and* newly constructed during these phases.

As to teaching, *DNR* articulates three foundational instructional principles: *duality, necessity,* and *repeated reasoning*.

The *Duality Principle*:

1. **Students develop ways of thinking through the production of ways of understanding, and, conversely:**
2. **The ways of understanding they produce are impacted by the ways of thinking they possess.**

The *Necessity Principle*:

For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' here refers to *intellectual need*.

The *Repeated Reasoning Principle:*

Students must practice reasoning in order to internalize ways of understanding and ways of thinking.

Part IV:

In this part of the talk, I will outline an authentic mathematical lesson guided by the *DNR* framework. The goal is to give the audience a concrete image of this framework and of its application in mathematics instruction. The lesson was conducted several times, some with inservice secondary teachers in professional development institutes and some with prospective secondary teachers in an elective class in their major. In the discussion of this lesson, they are referred to as students. The lesson will be described as a sequence of four segments of students' responses and teacher's actions. Each segment is further divided into numbered fragments to allow for reference in the analysis that follows. The first segment describes the problem around which the lesson was organized. For this talk to have its intended educational effect, the audience will be asked to solve this problem before proceeding to discuss the subsequent segments.