The role of a specific semiotic tool in the study of the systems of numeration and metric system (2nd and 3rd grade in France)

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Comparing lessons about systems of numeration (and metric system) between two periods: before the reform of modern mathematics and nowadays, we can notice important differences in mathematic rules (or properties) which are expressed in textbooks. Semiotic tools seem to have been about entirely renewed. What do these changes precisely consist of? Are they likely to have an impact on students practices and learning?

Theoritical frame

We use tools from a french theoritical frame which is called “anthropological theory of didactics” and “didactic transposition” (Bosch & Chevalard, 1999). These tools are notably fashioned to study “ecology of knowledges”. This means: how teaching objects are living, how they uprise, who they live with, how they die, etc.

In this paper, I use two tools of this theoritical frame:

- a *praxeology* describes a human practice. It is constituted by four pieces: a set of similar problems (called *a type of tasks*), a way to work the whole type of task (called *a technique*), an explanation of how the technique works (called *a technology*), a *theory* which is able to legitimate the technology,
- *ostensive objects* are semiotic tools (Bosch, 1994). An “ostensive object” has a “power of handling”. For instance, if in a given institution the notation $x^{\frac{1}{2}}$ doesn’t exist but only $\sqrt{x}$, you have to use the specific formula $\frac{1}{2\sqrt{x}}$ if you want to obtain the derivative of $\sqrt{x}$, whereas you can use a more general formula nx^{n-1} and apply it with $n=\frac{1}{2}$ if $x^{\frac{1}{2}}$ exists.

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Methodology

In order to study the beginning of the generalization of the systems of numeration (3 or 4 digits numbers) and metric system, our material is french textbooks of primary school from 2nd and 3rd grades and theoritical texts for mathematics at school. We search techniques and technologies to study the same types of tasks in multidigit numbers (and metric system) in old times[2] and nowadays, with a specific interest in ostensive objects.

Results

Preliminary comments

We begin with presenting french system of number words. Like in many other european langages, french langage has many irregularities with number words.

un (une) 1	deux 2	trois 3	quatre 4	cinq 5	six 6	sept 7	huit 8	neuf 9	dix 10
onze 11	douze 12	treize 13	quatorze 14	quinze 15	seize 16	dix-sept 17	dix-huit 18	dix-neuf 19	vingt 20
dix 10	vingt 20	trente 30	quarante 40	cinquante 50	soixante 60	soixante-dix 70	quatre-vingt 80	quatre-vingt-dix 90	cent 100
cent 100	deux cents 200	trois cents 300	400	500	600	700	800	900	1000
mille 1000	deux mille 2000	trois mille 3000	4000	5000	6000	7000	8000	9000	10000

We put the less irregular number words in yellow and the most ones in red. In pink, these are grammatical irregularities: arriving to one hundred when counting by one, french people don't say something like "98, 99, one hundred, one hundred and one..., deux hundred", but : "98, 99, cent, cent un, ... deux cents". The word "un" is missing before "cent".

We have another type of irregularities which are the words for what the call "units of counting". In english, you have "ten" and "a ten", then "twenty" is also "two tens". In french, we don't use the same word to say "ten" and a "ten". Words for units of counting with corresponding number words are:

french		english
number words	units of counting	words for both
un (une)	unité	one
dix	dizaine	ten
cent	centaine	hundred
mille	millier or mille	thousand

[2] "Old times" means here "before the reform of modern mathematics" (1970)

So, we have: une unité (one one), une dizaine=dix unités (a ten=ten ones), une centaine=cent unités (a hundred=one hundred ones), un millier=mille unités (a thousand= one thousand ones), we can also say : “un mille” for “un millier”, so un mille=mille unités (regular).

So when you count by ten for instance, you can count ones or tens, the result is:

- dix, vingt, trente, quarante, cinquante, etc. (ten, twenty, thirty, forty, fifty, etc.) counting ones by ten, or:
- une dizaine, deux dizaines, trois dizaines, quatre dizaines, cinq dizaines, etc. (one ten, two tens, three tens, four tens, five tens, etc.) counting tens.

Last, in french it is the same word to say “one(s)” and “unit(s)”, the word is “unité(s)”.

Ostensive objects for systems of numeration

Before the reform of modern mathematics, in France, when studying textbooks, we find five technologies which are organized around the same ostensive object. I begin with presenting the ostensive object, then the technologies.

We already have two wellknown ways to express numbers:

- system of place-value (or positional system of numeration) : *456*
- system of number words: *four hundred fifty six*.

We say these to systems are two ostensive objects to express numbers. But we have a third one, we call it: “system of units of counting” (or “system of units”). We describe it now:

- this ostensive consists in expressing number with two kinds of words: units like ones, tens, hundreds, thousands and other number words (from one to nine for the beginning). So for instance, we have the number *3 thousands 5tens*. (We consider it is the same ostensive object if we write : *three thousands five tens.*)
- this ostensive object is a regularization for the system of number words. For instance: “5 tens” for “fifty” (or “5 dizaines” for “cinquante”), but not only,
- indeed, the power of handling of this ostensive object is greater than the one of the system of number words. We can say : *56 hundreds*, *thirty tens* though there are no similar words in the system of number words.

Old technologies

Lets’s come now to the five old technologies. They are:

1. counting units: “One count by ten (hundred, ones of thousands) like one count by one”.
2. Relationship between units of counting: one hundred=10tens, one hundred=100ones.
3. Relationship between number words and units of counting (this is a translation word for word): trois cents=3 centaines (three hundred = 3 hundreds), cinquante=5dizaines (fifty=5tens), then trois cent cinquante=3centaines5dizaines (three hundred fifty =3hundreds 5tens)
4. Relationship between metrical units and units of counting (it is also a translation word for word): 1hectomètre=1centaine de mètres (1hectometre=1hundred of metre), 1hm=1 centaine de mètres, 1décagramme=1dizaine de grammes (1decagram=1ten of gram)
5. Relationship between place-value and units of counting : “in a number (of three digits), hundreds fill the third place from the right.” (“The digit 0 (zero) doesn’t represent any unit, it locates a place”).

We now take two very common examples of types of tasks to show how these technologies work: the first one is “let be a collection of things, what is its number?” (or reciprocal task), the second one is “let be a number in system of number words, write it in system of place-value” (or reciprocal task).

Supposing blocks of thousands, hundreds, tens and remaining ones are already done (less than 9 of each). With 1st technology you can count how many unit of each kind you have, for instance 3 thousands, 4 tens, 5 ones. Then use the 5th technology to obtain the positional written number: 3 thousands means 3 in 4th place, 4 tens means 4 in 2nd place, 5 ones means 5 in 1st place. The written number is 3045. If you want the number word you use 3rd technology instead of the 5th.

Supposing now, we want to know how to say 6507. First use 5th technology: 7 in the 1st place is 7 ones (7 unités), 5 in the 3rd place is 5 hundreds (5 centaines), 6 in 4th place 6 thousands (6 milliers). Then use 3rd technology: 7 unités is sept, 5 centaines is cinq cents, 6 milliers is six mille. Then (after using an implicit technology in order to place units from bigger to smaller), you can say 6507 is six mille cinq cent sept (six thousand five hundred seven).

In fact, we could enunciate some technique for the 2nd type of task for instance, very near from the technology. In old books, we can find it sometimes but not very often: “to read a 3 digit number, one read first the digit of hundreds, then the number formed of the two other digits."

In fact, our ostensive object "system of counting by units" is an intermediate ostensive for about any type of task in learning numbers.

How is it today?

We have clearly lost the homogeneity of the old books, so the situation is more difficult to study (and to describe). Anyway, none of the books we have read uses the technologies we describe to find the number of a collection. Furthermore the "system of units" seems to have been replace by another ostensive object that we call "system of additive writings". We give examples from four books. They do not do the same way. We only consider the first type of tasks.

Let's see the first textbook (2003) [1]. We have 2 blocks of 1000, 4 blocks of 100, 3 blocks of 10 and 7 no grouping objects, teacher have to write 1000, 100, 10 on the black board. Then student have to transform these into 2000, 400, 30 and 7 (we don't know how they do, but probably counting ones by 1000, 100, 10 using an algorithm with words or positional system like "1000, 2000", then "100, 200, 300, 400", then "10, 20, 30"). This is a lack of the 1st technology. Yet, they write 2000+400+30+7 (or 2000+30+400+7). The problem is

> "how to write the number which shows how many objects there are?"

After studying propositions of students, the answer to the question is given:

> "2, written on the left of the number, rates 2000 ones; that is what the 2 blocks of 1000 contain; we use to say 2 is the digit of thousands", make student write 1000 on each block. "4, written at the right of the digit of thousands, rates 400 ones; that is what the 4 blocks of 100 contain; we use to say 4 is the digit of hundreds", make student write 100 on each block. (...)"

The former edition (1995) [2] of this textbook used another way, it put numbers (1000, 1000, 500) one under the other (just like in a column addition) and write the result. It was not using any word of the "system of units". Another book (2002) [3], with a very less socio-constructivist way, gives the solution. It consists in writing:

1000×2	+	100×…	+	10×…
↓		↓		↓
2000	+	…	+	…

= …

Another textbook (2004) [4] do another way. Student have to find the numbers of « 1 box of 1000 et 5 boxes of 10 » (pencils). It uses a kind of array (very common in France):

thousands	hundreds	tens	ones

> "[Students] write numbers in the array."

Then teacher says:

> "We write a zero in the columns where there is nothing. For this writing we come back to the canonic decomposition:
>
> 1050=(1×1000)+(5×10)
>
> 1050=1000+50. »

It seems that student can obtain the number using the array but to justify it they have to calculate. To sum up: the technique is to do the number in the array with a small use of "system of units" because the units remain in the array, its "power of handling" is very reduced. The technology is to calculate with rules we can't find at this time of the book (like in all the book we evoked except in [1] where they say, result of the addition is a convention).

Conclusion

We show that there is at the present time a big heterogeneity in technologies when studying system of place value. There are differences between textbooks and also among a given textbook when we cross technologies for different tasks (not shown here).

We have a "new" ostensive object we call "system of additive writings". In the textbooks, it is not so clear how to use it: calculate with it or apply it the "convention of position"? It seems it depends on the textbooks but some of them don't say anything. It seems not to be a good intermediate ostensive object.

All these variations seem to be the consequence of the removal of the old ostensive object "system of units". Furthermore, we also see it coming back in techniques but not in technologies (not yet), like in [1]. We think all this shows that "system of units" is missing in present technologies and that the "system of the additive writing" cannot replace it.

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