Games of interpretation, anticipating thought and coordination between verbal and algebraic register: key-aspects in the analysis of students' proofs in elementary number theory

Annalisa Cusi
Dipartimento di Matematica - Università di Modena & Reggio E.

Abstract

This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense through teaching experiments on proof in elementary number theory (ENT). Our aim is to analyze the use and the role of algebraic language in the development of such proofs. In this paper we present the analysis and classification of students' behaviour in facing the proof of a conjecture while working in small groups. The analysis of students' protocols was made by reference to the following interpretative-keys: the application of specific conceptual frames, the games of interpretation between different frames, anticipatory thoughts, the use of conversions and treatments and coordination between different registers of representation. Our analysis highlights the incidence of anticipatory thoughts and of the flexibility in the coordination between different frames and different registers of representation in development of proof in ENT.

1. Introduction

Many research studies support an approach to algebraic language related to the development of reasoning. Arcavi (1994, 2005) claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, expressing and communicating generalizations, the establishment of connections, or the production of argumentation and proof. Bell (1996) describes "the essential algebraic cycle" as an alternation of three main typologies of algebraic activity: representing, manipulating and interpreting. He states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects. We adopt Wheeler's idea that activities of proof construction could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". We believe that activities of proof in ENT would not only provide students with the opportunities they need to progress gradually from argumentation to proof, but would also help them to appreciate the value of algebraic language as a tool for the representation and solving of situations that are difficult to manage through natural language only (Malara 2002, Selden & Selden 2002). In order to foster the diffusion of activities of proof in ENT in school mathematics, we are working both with pre and in-service teachers (Cusi & Malara 2007) and with students. The activities we focus on in this paper refer to students and are part of a long-term experimentation realized in some classes (10^{th} grade) of a Liceo Psico-Pedagogico.

1.1 Methodology of work with students

Aiming at making student acquire an effective symbol sense, we planned and experimented a path for the introduction of proofs in ENT through different gradual phases of work. The path can be subdivided in 6 main phases, characterized by the following activities: (1) Translations from verbal to algebraic language and vice-versa; (2) Study of the relationship between properties of a given formula and properties of the variables it contains; (3) Analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) Exploration of numerical situations, formulation of conjectures and related proofs; (5) Construction of proofs of given theorems. The path (about 20 hours) was articulated through small-groups activities (8 groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. In this work we will dwell on a central moment in the path: the small-groups' work aimed at constructing the proof of some conjectures they produced starting from numerical explorations. In particular, we will present the main results of our analysis of group discussions when students were trying to prove one of the conjectures.

2. Theoretical framework which support our analysis of students' discussions

We identified a set of theoretical references that are both appropriate to the analysis of the transcripts of group discussions dealing with proofs, and in tune with the view of algebra that we are promoting. The main reference in our research is the work by Arzarello *et Al.* (1994, 2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient *tool for thinking*. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem", and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes. According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. Boero defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process". In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that we take as fundamental for analysing students' management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems "that permit a transformation of representations". Among them, he includes both natural and algebraic language. Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object's representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations. He distinguishes between *treatments* ("transformations of representations that happen within the same register") and *conversions* ("transformations of representation that consist of changing a register without changing the objects being denoted"), highlighting both the fundamental role of each of these typologies of transformations and the intertwining between them.

3. Research hypothesis and purposes

Our hypothesis is that the production of good proofs in ENT depends upon the management of three main components: (a) the appropriate application of frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels). Our aim is to investigate the following aspects: (1) Effectiveness of the theoretical references we selected as tools for analyzing and classifying students discussion about proofs in elementary number theory; (2) Identification of the essential components for good productions in this context; and (3) The role played, by the three components we singled out (application and coordination between frames, anticipating thoughts, coordination between verbal and algebraic register) and the mutual relationships between them. In this work we will present a sample of prototype-productions helpful to verify our hypothesis and to highlight that the lack or unsuccessfully application of one of these components leads to failure and/or blocks of various types.

4. Research Methodology

Theoretical models we used helped us to identify some interpretative keys for both the analysis of protocols and their subsequent classification. Our analysis focused on the following: (1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between the different frames appropriate to those same expressions; (2) The application of anticipating thoughts; and (3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

4.1 Analysis of small-groups activities

The study of small-groups work is proposed by researchers with different aims: some of them are interested in the effects of these kind of activities as instruments to promote learning (Barnes, 2005), other researchers focus on the dynamics which characterize the students' mathematical discourses while they are working in groups (Ryve 2006), others aim at highlighting how individuals re-construct mathematical concepts through small-groups interaction (Vidakovic & Martin, 2004). Our choice of making students work in small groups is, instead, motivated by a different reason. Our conviction is that only when students are involved in a communication it is really possible for us to produce an in-depth analysis of the coordination between verbal and algebraic register in the construction of proofs in ENT. We believe that the analysis of the sole written protocols is not enough to highlight students' abilities in carrying out correct conversions from verbal to algebraic language and their actual interpretations of the constructed algebraic expressions. The need to communicate their reasoning to others forces students not only to verbally make what they are writing explicit, but also to explain both the objectives of the transformations they carry out and their interpretation of results.

5. The problem and its *a priori* analysis

The problem we posed to students is the following: *"Write down a two digit number. Write down the number that you get when you invert the digits. Write down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state"*.

The regularity to be observed is that the difference between the two numbers is always a multiple of 9 where the multiple is the difference between the digits of the first number. The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$.

The initial conceptual frames to which the statement of the problem refers are 'difference between numbers' and 'two digits numbers'. It can be assumed, therefore, that the student will not automatically choose the 'polynomial notation' frame to represent the problem and apply the necessary simple treatments to make the conjectured property explicit (some students might apply the 'positional representation of a number' frame and then get stuck). The reference to the 'divisibility' frame, which allows them to foresee the desired final shape of the expression after correct treatments (i.e. $9 \cdot k$, where k is a natural number), is, instead, less problematic. Possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the final shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame 'divisibility'). Finally, we make some observations about possible student behaviour. Many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers. Consequently, the analysis of the final expression could provide another index of a students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

6. The analysis of a prototype-production

In the analysis of small group work, we singled out the incidence and the interrelation between the following: (a) the application of and coordination between frames, (b) ability in the game of interpretation required to produce the proof, (c) display of appropriate anticipating thoughts, (d) ability to correctly perform treatments and conversions, and (e) ability to coordinate verbal and algebraic registers. In this paragraph we will present our analysis of a problematic protocol. We chose it because it better highlights how students' interaction allows to identify the reasons of erroneous conversions and the difficulties in the interpretation of expressions. The following transcript documents the application of a suitable frame, associated to an inadequate conversion and to the incorrect interpretation of the produced expressions.

The Protocol

After having considered many numerical examples, students A, C and N conclude that the considered difference is always a multiple of 9. The following dialog represents the proving phase.

C (27) : Let us do with letters.
N (28) : It is more complicated.
C (29) : It will be $10x$ … plus …
A (30) : …plus y *(they write* $10x+y$*)*[1]
C (31) : If we invert the digits, it will be $y+10x$
N (32) : Yes.
A (33) : and now … we have to do the difference
C (34) : (*she writes and reads*) $10x+y$
A (35) : Let us put it in brackets
C (36) : minus ... ($y+10x$)
C (37) : it becomes $10x+y-y-10x$
N (38) : I think there is a mistake because the result is zero …
A (39) : It become $10x+10x$?
N (40) : No, they cancel each other out.
C (41) : Meanwhile, let us write: (*she dictates*) they are all multiple of 9 … It is not simple …
N (42) : We are not able to prove it. It is difficult.
C (43) : We have 10x+y and it represents the number … Then we have to …
A (44) : *(she reads)* 'when you invert the digits' …
C (45) : It is the same of having 1 and …
C (46) : It is as if we take it on this side, so *y* should be take on the other side.
C (47) : however, if we take 10 on this side, it will be left a …
A-N-C (48) : one!
C (49) : So it is not 10x. I think it is x …
A (50) : Let us try!
C (51) : So it would become $10x+y-(y+x)$. The two *y* cancel each other out, so they will be left $10x-x$. Exactly $9x$! We were able to prove it!
C (52) : (*C. is looking to the numerical examples*) But here I can see something more, I think. I can see that, practically, this is … Look what I noticed (*she is looking at the differences 86-68, 92-29, 76-67, 52-25*) … if you subtract the two tens, 8-6, you have only to consider the product between 9 and the difference

[1] The difficulties we hypothesised in the identification of the initial frame are not highlighted by this protocol because students have faced the problem of the representation of two and three-digit numbers in a previous activity.

between the two tens: 9 times 2 is 18; 7-6 is 1, 9 times 1 is 9; 5-2 is 3, 9 times 3 is 27. A (53) : We have to write it down. I would have never noticed it! C (54) : (*she dictates*) It is always a multiple of 9 and we can observe that the result of the subtraction … you have to subtract the two tens and to multiply the result by 9. C (55) : Do you know how I though of it? Because I saw 9x and I said "it is a multiple" because there is 9 times x. Then I said "but … what is x? x is the tens!". Then I tried to do x minus x. A+N (56): Good!

This protocol can be subdivide in three key-moments: (1) *Initial conversion and first treatments* (lines 27-37); (2) *Identification of a problem, modification of the conversion and new treatments* (lines 38-51); (3) *Attempt of interpretation of the obtained expression and refinement of the conjecture* (lines 52-56). The protocol highlights, first of all, the central problem in groups' works on this proof, that is the difficulties students met in representing the number obtained inverting the two digits. Success in this conversion requires both a good coordination between the 'positional notation' and 'polynomial notation' frames and a complete internalization of the last. Initially C. carries out a *first erroneous conversion* (line 31), translating this concept through the expression $y+10x$ (she only changes the order of the addends). The students correctly interpret the natural language term "invert" when they work on numerical examples in order to formulate the conjecture. Afterwards, however, when they have to carry out a conversion into algebraic register, the concept "exchanging the place" is translated through the pure exchange of the order of the monomials which constitute the polynomial $10x+y$. The difference (zero) they obtain starting from this erroneous conversion lead them to detect the inaccuracy of their initial conversion and to look for a new correct one. They detect a mistake in having supposed that $10x$ should represent the units digit (only because it is on the right side of the polynomial $y+10x$). So they decide to correct this mistake, substituting x instead of $10x$, but they do not consequently modify the representation of y as tens-digit. Therefore, writing the polynomial as $y+x$, they carry out again an incorrect conversion. Probably because of the prevailing of the anticipating thought they carry out (expecting a multiple of 9, they only concentrate on the factor 9 when they look at the expression $9x$), once they obtain $9x$ as the difference between the two numbers, they do not immediately subject the new result to a careful interpretation. Only afterwards C. interpret x as the tens-digit of the initial number and decide to investigate the considered examples in order to refine their conjecture. C. concentrates on the tens-digits of the two numbers (x and y in the correct representation) and observes, starting from examples, that the result is obtained multiplying 9 by the difference between those digits. This observation, however, does not help her in critically interpreting the expression $9x$. In her final intervention, she even tries to translate into algebraic language, through the expression x-x, the difference between the two tens, but she is not able to 'grasp' the gap between the algebraic representation she proposes and her verbal considerations.

7. Conclusions

The protocol we proposed highlights the strict correlation between lack of flexibility in coordinating different frames, difficulties in carrying out conversions from verbal to algebraic register and lack of interpretative games in the analysis of the expressions produced. Moreover, it testifies how such correlation causes failures in the production of proofs in ENT. In fact, the three students display rigidity in their use of frames and an incapability of simultaneously manage different frames. Such rigidity make them produce partial or incomplete interpretations of the constructed expressions, so they are not alerted about the non-acceptability of their proof. Because of space limitations, we cannot dwell on the analysis of other protocols, but we would like to share some of the main conclusions we took. The rigidities highlighted in the analyzed protocol are shared by other protocols, to which different problems could be add: (a) blocks in the treatments and in the interpretative processes due to an inability to foresee the expression to be attained by the activation of the correct final frame; (b) difficulties in the choice of the treatments to be operated caused by the absence of anticipatory thoughts; and (c) interrelation between 'blind' manipulations (i.e. produced without a conjecture, therefore without an objective) and blocks in the interpretative processes. Through our analysis we were able to highlight four main categories of prototype-productions: (C1) Partial reference to algebraic language and presence of blocks; (C2) Application of a suitable frame, but inadequate conversion and incorrect interpretation of the produced expressions; (C3) Appropriate application of frame and conversion, not supported by semantic control and anticipating thoughts; (C4) Good coordination between frames and good interpretation of the expressions in the applied frames. The analysis and the classification we made also allow us to verify the effectiveness of the theoretical elements we selected as tools for analyzing small-groups' discussions for the construction of proofs in ENT. The protocols we classified in C4 represent an evidence of the fact that an appropriate application and combination of the three components we highlighted is a necessary condition for the proper development of a proof in ENT. That is: appropriate application of frames and appropriate coordination between them; appropriate anticipating thoughts; appropriate coordination between algebraic and verbal registers.

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