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Approaches to the Integral Concept

The case of high school calculus

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Abstract

This study is motivated by a very real problem in the high school mathematics classroom: High ability students have great difficulties with the integral concept. The students do not acquire comprehension regarding the concept of the integral and are satisfied, in the best case, by formal techniques to the solution of the problems.

I propose to investigate to what extent this can be attributed to the way integrals are normally introduced in high school. For this purpose it will be necessary to conceptualize the meaning of comprehension; to investigate the relation between the inner structure of the integral concept, and comprehension of the concept; to devise and examine coherent didactical tools for developing significant concept comprehension in learners; and to investigate which types of problems relate to the inner structure of the integral concept.

According to my hypothesis, the mathematical idea that enables a deeper comprehension of the integral and doesn't damage skills is to introduce the integral as an accumulation function (Thompson, 1994; Thompson & Silverman, 2007).

From here, the main goals of the current study can be formulated in the following way:

- To analyze how the theoretical (didactic and instructional) components of the integral concept are connected to the deep comprehension of the concept

- According to the results of this analysis, to analyze a common approach to the concept of the integral and the concept of accumulation function,

- To evaluate the present situation concerning students' knowledge about the integral concept,

- To develop a unit of instruction on the topic of the integral that is based on the initial presentation of the concept as an accumulation function and to examine, mainly qualitatively, the progress of students regarding both, the technical skills and deeper comprehension.

This research, based on theoretical considerations and on empirical evidence, may shed light on the learning of integral calculus in high school and point to ways of improving it.

Learning the concept of the integral is an **important** part of the high school mathematics curriculum in almost the entire world, and among others also in Israel. In my eyes, this **importance** is absolutely justified: It isn't possible to imagine modern scientific culture without integrals. This concept (along with its relative, the derivative) constitutes a mathematical domain that is a language, a device, and a useful tool that is very important for other fields: physics, engineering, economy, and statistics. This domain is usually known as differential and integral calculus. Along with this, the concept of the integral represents a philosophical idea for the understanding of the world: contemplation of the totality of little parts of a whole enables conclusions regarding the whole in its entirety, as well as its internal structure and properties.

It is important to point out that the idea of integral was created and grew from within physics, from within the attempt to invent a mathematical tool that enables people to describe, to analyze and to explain different physical phenomena, e.g. volume, mass, work (Newton, 1686).

The most prevalent approach in schools (as well as colleges and universities) to the instruction of the integral is based on the definition of the indefinite integral as an inverse operator to the derivative (antiderivative). From here, one arrives at the definite integral with the help of different formulations of the Fundamental Theorem of Calculus (Spivak, 1967; Goren, 2005; Aspis, 2006; Meytav, 2006; Yakuel, 2006). Based on this approach students acquire (or, at least, are supposed to acquire) technical skills for the treatment of classical problems using integrals: computation of areas (and less frequently, volumes) of shapes (bodies) whose boundaries are defined by means of graphs of some elementary functions (whose antiderivatives are also elementary functions). In this way, do the students also acquire or develop the comprehension of the concept more deeply? Professional literature (Orton, 1983; Ashkinuze, 1987; Thomas & Hong, 1996; Sealey, 2006) and my personal experience show that most of the students do not acquire comprehension regarding the concept of the integral and are satisfied, in the best case, by formal techniques to the solution of the problems.

This begs the question whether another approach to the instruction of the integral concept exist, which enables, on one hand, not to lose technical skills, and on the other hand, to develop deeper comprehension of the concept? My hypothesis is: Yes, such an approach exists. This question is made up of two facets:

- Theoretical aspect (possible potential of the proposed approach): Is it correct that students can acquire comprehension of the concept by a certain approach?
- Empirical aspect (effectiveness of proposed approach): Do students, who have been exposed to the proposed approach, indeed acquire the expected comprehension?

Before specifying a hypothesis about the nature of such an approach, one should ask whether such an approach aiming at deeper comprehension is really needed. The study proposed here is based on a central proposition: Significant comprehension of mathematical concepts is at the heart of the instruction of mathematics. Therefore every effort to develop comprehension is wanted and required. This proposition becomes almost evident when considering integrals. The idea of calculus in general and the idea of integral in particular were born from human attempts to understand the world, from applications (Newton, 1686). In some way the integral is the application. So, in my eyes, there is no way to understand such an idea without understanding the strong connection between the mathematical concept and its application, in other words, without significant comprehension.

In undergraduate education, usually the same approach is used as in school; there are, however, a small number of exceptions: Initial presentation of the concept on the basis of Riemann sums (Taylor, 1992; Courant, 1934; Apostol, 1969), on the basis of average height (Turegano, 1998), and on the basis of differential equations (שטיין, 1995). Indeed, in high schools, any presentation of the integral other than the common one is extremely rare. However, even superficial contemplation on the common approach raises doubts:

- If integral "is" antiderivative – it is easy (even at a high school level) to bring an example for a function, the integral of which cannot be recorded with elementary functions. So in this case, for high school students, the integral is unknown and might be considered non-existent…
- If integral "is" area - it seems difficult to deal with situations where integral is volume, and such situations occur even in high school…
- If integral "is" limit – it is difficult to overcome high school students' lack of familiarity with the concept of limit…
- In contrast to the concept of the derivative, for which a language that is rather rich and varied is developed in schools, regarding the concept of the integral the language is amazingly poor.
- And apparently, there is simply no place for the use of preliminary intuitions (Fischbein, 1978; Fischbein, 1987) and it is problematic to identify and to operate cognitive processes that are unconnected and independent (like, for example, those that were described in the theory of **procept** (Gray & Tall, 1994)) that lead to deeper comprehension of the concept.

My hypothesis is that the mathematical idea that enables a deeper comprehension of integral and doesn't damage skills is to consider the integral as an accumulation function (Thompson, 1994; Thompson & Silverman, 2007), in the plain meaning: a sum that has a large number of very small terms.

There are at least three reasons which strengthen this hypothesis:

- The idea of accumulation allows in a natural way to combine the concepts of definite and indefinite integral and its connection with the concept of the derivative.
- The idea of accumulation allows in a natural way to represent the connection between the mathematical idea of integral and its applications.
- The idea of accumulation allows in a natural way to represent the generalization of the mathematical concept of integral, e. g. to Riemann-Stieltjes integrals, Lebesgue integrals, Lebesgue-Stieltjes integrals, and Denjoy (gauge) integrals.

I think, and I hope to show that the idea of accumulation can make use of students' intuitions and enable them to develop and to operate cognitive processes that will lead to deeper and perhaps wider comprehension of the concept of integral.

From here, the main goals of the proposed study can be formulated in the following way:

- To analyze how the theoretical components of the concept of the integral are connected to the deep comprehension of the concept;
- According to the results of this analysis, to analyze a common approach to the concept of the integral and the concept of accumulation function;
- To design and to offer a unit of instruction for the integral concept based on the idea of accumulation function;
- To examine, mainly on the qualitative level, the progress of students learning with this unit regarding technical skills as well as deeper comprehension.

It is important to notice the following:

- Analyzing the comprehension of the integral concepts requires the conceptualization of the central proposition regarding the concept of the integral;
- Analyzing a common approach to the concept of the integral and accumulation requires the testing of the present situation: students' understanding regarding the concept of the integral;
- Analyzing the concept of accumulation function requires the mathematical consolidation of this concept.

I hope that the results of this study will give mathematics teachers and their students an additional, perhaps more efficient, possibility to acquaint themselves with, to learn, and to understand one of the most brilliant mathematical philosophical ideas - the concept of the integral.

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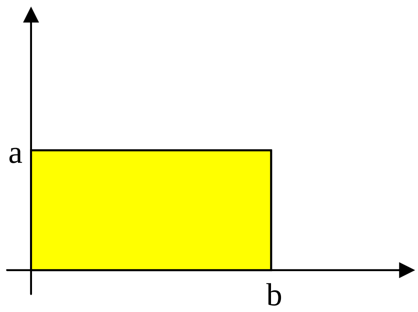
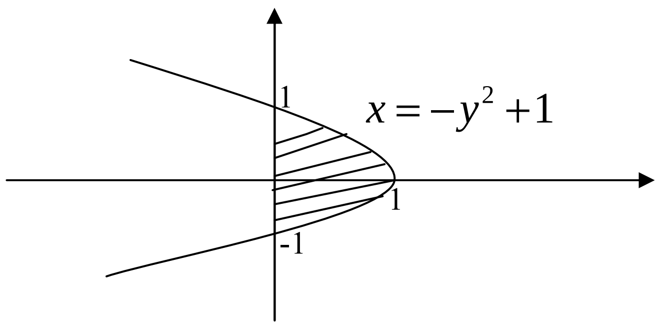
Appendix

1. (i) Write an integral allowing to calculate the area of the rectangle given in the figure.
(ii) Calculate the area of the rectangle with the help of the integral that you wrote in (i).



2. If it is possible, calculate the indicated area. If not, explain why it is impossible:



3. Compare the results of the following integrals and explain your findings.

a. $\int_{-1}^{0} x^2 dx$ and $\int_{0}^{2} x^2 dx$

b. $\int_{-2}^{3} x^3 dx$ and $\int_{-1}^{4} (x-1)^3 dx$

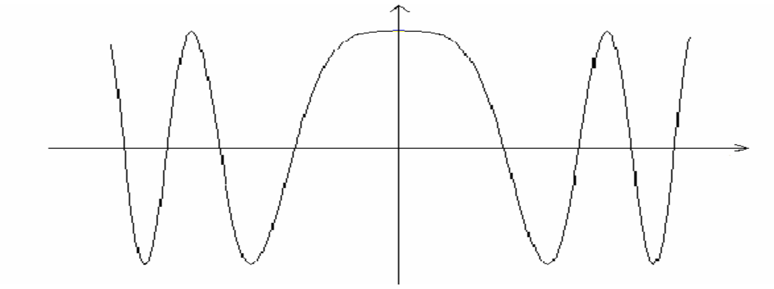
c. $\int_{0.01}^{100.01} x^2 dx$ and $\int_{0.01}^{100.01} (x^2+1) dx$

d. $\int_{-2.17}^{1.23} \pi x^4 dx$ and $\int_{0}^{2.17} \pi x^4 dx$

4. Does the integral $\int \sin x^2 dx$ exist? Explain your answer.

5. Consider the sketch of the graph of the function $y = \cos x^2$ given in the figure.

   Does the integral $\int \cos x^2 dx$ exist? Explain your answer.



6. If $\int_{-3}^{5} f(x)dx = 10$ then $\int_{0}^{8} f(x-3)dx = ?$. Explain your answer.

7. If $\int_{1}^{50} f(x)dx = 1$ then $\int_{1}^{50} (f(x)+2)dx = ?$. Explain your answer.

8. If $\int_{-5}^{5} f(x^4)dx = 11$ then $\int_{-5}^{0} f(x^4)dx = ?$. Explain your answer.